Statically Indeterminate Structures

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Lawrence C. Maugh, PROFESSOR OF CIVIL ENGINEERING
UNIVERSITY OF MICHIGAN

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Preface to Second Edition

In this second edition practically all original material has been rewritten and new examples and problems included. Chapters 2 and 3 replace the original Chapter 11 on the basic principles of structural mechanics. This subject matter has been expanded, especially in the field of virtual work and angle changes. Both old and new material anticipate the increasing use of digital computers which have already proven of great value in the solution of structural problems.

Chapters 4, 5, 6, and 7 emphasize the practical application of the slope-deflection and moment-distribution methods. Additional material has been added to the analysis of continuous frames subjected to lateral forces as well as new problems and examples.

Although Chapters 8 and 9 have been rewritten, the original presentation of the methods of analysis for continuous trusses, bents, and arches has been largely retained. I believe that the methods described provide the most practical solutions for the structural designer.

The subject matter in Chapter 10 is considerably expanded and revised. New material on catenary cables and guyed towers is included together with additional examples of both tower and suspension bridge analysis. Many new problems have been provided to assist the reader in testing his understanding of the subject.

Chapter 11, which replaces the original Chapter 10, has been retained essentially in its previous form. Since I have found this material to be

valuable in design work, even though it is not usually included in most structural courses, I feel that it should be available to the reader.

I wish to thank my many friends who have contributed ideas, advice, and criticism. My associates at The University of Michigan and Professor Donald L. Dean of the University of Delaware deserve special mention.

L. C. MAUGH

November 1963

Preface to First Edition

The contents of this book have been selected largely from material developed for courses in the analysis of statically indeterminate structures for senior and graduate students at the University of Michigan. The methods of analysis that are explained and illustrated are based on fundamental principles of structural mechanics that are applicable to the design of most frame structures. In the application of the fundamental principles, particular emphasis is accorded to numerical solutions by various methods of successive approximations, such as moment distribution, iteration, trigonometric series, and the panel method. The analysis of indeterminate structures by means of successive approximations instead of by the more laborious methods that require the solution of many simultaneous equations is undoubtedly the most notable advancement in structural design in the past two decades.

Although the numerical work may be performed by methods of successive approximations, the student must be able to express the relationship of the various physical factors in both geometric and algebraic form. In other words he must thoroughly understand the various methods for calculating displacements in all types of structures and be proficient in their use. This work will naturally precede the study of deformation equations, in which the redundant forces are expressed in terms of displacements. Once the student is familiar with the deformation equations, the principle of superposition, and the general reciprocal

theorem, he should have no great difficulty in developing a facility in their application which will enable him to solve most structural problems.

I realize that practically all the fundamental principles that have been employed in this book were developed years ago by engineers to whom the engineering profession is greatly indebted. Among those who should be mentioned particularly are Maxwell, Mohr, Müller-Breslau Ostenfeld, Castigliano, and Williot. In addition to these earlier engineers, I also wish to acknowledge the influence of more recent work that has been contributed by Maney, Cross, Timoshenko, Southwell, and many others. The publications of these men have been listed in the references at the end of the various chapters, where they can easily be found. It is hoped that the student will use them.

In the arrangement of the subject matter and in checking the numerical solutions, much valuable criticism and assistance have been given by former students, particularly C. W. Pan, G. Hoenke, and P. C. Hu. I wish to express my appreciation to the Portland Cement Association for permission to use the diagrams on pages 321 to 326, to Mr. D. S. Ling for the diagrams on pages 327 to 331, and to many friends and associates for their help and advice.

L. C. MAUGH

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Contents

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Classification and Description of Statically Indeterminate Structures

1.1 Equilibrium Conditions

The classification of structures as statically determinate or indeterminate is usually expressed in mathematical form, although the physical character of the structural framing is the real factor. Structural framing means the arrangement and composition of the structural members, the number and characteristics of the supports, and the structural details of the connections. In any structural problem the designer must determine the boundary forces for each member or assemblage of members with due consideration to the restraint that the particular connection will provide. It is assumed that the reader is familiar with the physical laws that force systems must satisfy to maintain a structure in a state of static equilibrium that is, that coplanar force systems must have no unbalanced components in two arbitrary directions and no resultant moment about any point. These requirements are commonly expressed in the following convenient algebraic form.

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_{xy} = 0 \tag{1.1}$$

For space frames subjected to a three-dimensional force system, another direction of translation and two additional planes of rotation are possible, and consequently the equilibrium of such structures requires that both equations 1.1 and 1.2 be satisfied.

$$\sum F_z = 0 \qquad \sum M_{xz} = 0 \qquad \sum M_{yz} = 0 \tag{1.2}$$

Since the external forces applied to the structure are determined first, they will ordinarily constitute the constants of equations 1.1 and 1.2, whereas the reactive forces are the unknowns or variables. When the number of unknown forces is just sufficient to satisfy equations 1.1 and 1.2, that is, not more than three for coplanar force systems and not more than six for space frames, the structure is statically determinate. However, only two of the three coplanar forces can be parallel or intersect at a point, and for space frames, the directions of the reactions must be such that they cannot all be intersected by one straight line, otherwise there is not full restraint. When the number of unknown forces is more than is necessary to satisfy the conditions expressed by equations 1.1 and 1.2, the structure is described as statically indeterminate or hyperstatic. The surplus or excess forces are termed redundant forces, which, although unnecessary for equilibrium, may be desirable for other reasons. Since the application of equations 1.1 and 1.2 to statically determinate force systems should be familiar to the reader, these problems are not repeated here. Moreover, the use of these equations in the analysis of statically indeterminate force systems is treated hereafter as a necessary and fundamental part of the solution.

1.2 Discussion of Boundary Forces

The mathematical requirements as stated form a criterion to be used to establish the degree of indetermination after the number and nature of the external and internal forces have been determined. A careful study of the physical action of each support and connection is essential for a correct determination of the boundary forces that must be considered in the design.

To decide upon the nature of the reactive forces, accurate information must be obtained for the restraint to the motion of the structure given by each connection. This restraint can, in general, be classified as resistance to translation, to rotation, or to both. If the amount of this restraint is small, the reactive force produced will also be small and can frequently be neglected. The decision made at this point may actually decide whether the structure is to be considered statically determinate or indeterminate.

Thus, if the base of the column AB in Fig. 1.1a is assumed to rotate about point A, then the resultant force at A can be resolved into a horizontal component H_{AB} and a vertical component V_{AB}; although at times it may be more convenient to replace these quantities by the resultant force R_A and the angle β. In either form two variables are required in any structural analysis.

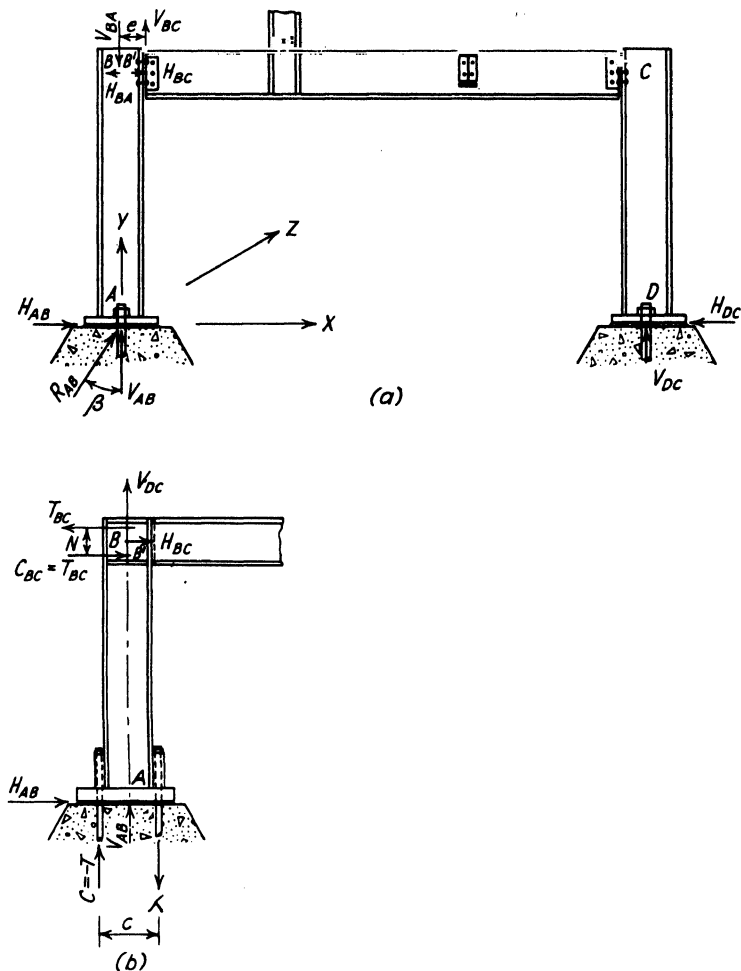



Fig. 1.1

When the base is arranged as shown in Fig. 1.1*b*, two equal and opposite forces T are also applied to the column. The moment of this couple is T_C which is designated by M_{AB}. In Fig. 1.1*b* two forces and a couple are applied to the column in the xy plane shown. In addition, although not shown, one force may act at A in the z direction as well as couples in the xz and yz planes, which would give a total of six unknowns. For practical reasons, the designer must frequently decide whether the forces in the xy plane are sufficiently independent of the action in the xz and yz planes as to reduce the analysis to one of three independent coplanar force systems. This assumption is frequently used for frame structures,

although it is not feasible for many types of slab, dome, and shell arrangements. For the present only coplanar force systems will be discussed.

Similarly, if the beam BC is connected to the columns as in Fig. 1.1*a*, the force acting at B in the xy plane can also be represented by two components, H_{BC} and V_{BC}. The eccentricity e is often neglected for the beam BC, that is, V_{BC} is assumed to act at B instead of B'. The assumption is on the safe side for the beam, but neglects the bending effect on the column. When the connection is arranged as shown in Fig. 1.1*b* the forces (couple) T_{BC} will also act and the moment $T_{BC}z$ can be designated by M_{BC}. In the present and future representation of boundary forces the subscript will indicate the section and the member upon which the force is acting. Thus H_{BC} indicates the horizontal component acting upon member BC at section B, and M_{CB} indicates the couple applied to the member BC at section C. Unless otherwise defined the axis of the member is assumed to pass through the centroid of the cross section.

1.3 *Idealized Force Systems and Stress Distribution*

Any discussion of idealized force systems and stress distribution may seem unnecessary to the student since that form is probably the most familiar to him. Nevertheless, even under the common arrangements shown in Figs. 1.1*a* and 1.1*b*, the actual stress and strain conditions in the material near the connections may be quite different from the linear variation that is commonly assumed. For the beam connection in Fig. 1.2, the distribution of the axial strain across section 1-1 is distinctly nonlinear; whereas at section 2-2, which is at least twice the depth of the beam from the connection, the distribution of strain and stress is practically linear across the depth of the section.

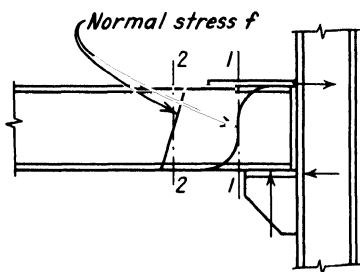



Fig. 1.2

When the end connections apply forces that are distributed across the entire end sections, a linear distribution across the interior sections is obtained more quickly. In other words, the more concentrated the boundary forces, the more nonlinear is the variation of the internal strain and the more likelihood of a local yielding and possible fracture of the material. Experimental studies may be required to obtain data for the solution of such strain problems.

For convenience the designer uses certain symbols and diagrammatic representations to indicate his opinion of the forces and restraints at the

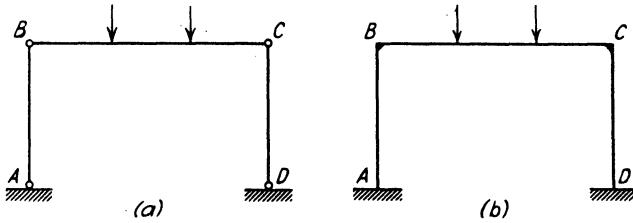



Fig. 1.3

boundaries. These symbols are shown on line diagrams that coincide with the axes of the various members. For example, the frame in Fig. 1.1*a* can be represented by the diagram in Fig. 1.3*a*, and the frame in Fig. 1.1*b* by the diagram in Fig. 1.3*b*. Difficulties will naturally arise whenever definite and rigid symbols replace rather indefinite boundary conditions; consequently, the student must adjust to these uncertainties by thinking of the actual as well as the idealized structure.

1.4 Selection of Redundant Forces

Once the characteristics of the boundary forces have been established, their number and arrangement can be studied with respect to fulfillment of the requirements for statical equilibrium as expressed by equations 1.1 and 1.2. If any boundary force or forces are removed, and the equilibrium of the structure or member is maintained by the remaining forces, the force or forces that are removed can be classified as redundant. The remaining boundary forces that are necessary to provide equilibrium are designated as reactive forces to distinguish them from the redundant forces. The statement that redundant forces may be removed, provided the means for statical equilibrium is maintained, is in itself a sufficient condition for the selection of redundant forces for most structural arrangements. Several examples of the selection of redundant forces and reactions will be given to illustrate the application of this definition.

Example 1.1 The arch in Fig. 1.4 is assumed to be restrained against both rotation and translation at supports *a* and *b*. Therefore both horizontal and vertical components will be applied as well as couples whose moments are M_a and M_b. A study of the equilibrium equations 1.1 will prove that the following combinations of forces can be removed and the equilibrium of the arch will still be maintained.

$$H_a, \quad V_a, \quad M_a$$

or

$$H_a, \quad M_a, \quad M_b$$

Other combinations of three redundant forces can also be used.

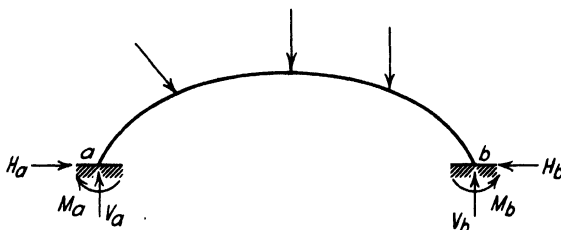



Fig. 1.4

If the following combination of forces are removed, the equilibrium of the structure cannot be maintained.

$$H_a, \quad H_b, \quad M_a$$

or

$$V_a, \quad V_b, \quad H_a$$

or

$$M_a, \quad M_b, \quad V_a$$

Therefore these combinations of forces cannot be selected as redundants.

Example 1.2 The frame in Fig. 1.5*a* is shown in the idealized form with hinge supports at *a* and *f*. The columns are continuous from *a* to *c* and from *f* to *d*. The member *be* is hinged to the columns, but *cd* is restrained at the joints *c* and *d*. The boundary forces at *a* and *f* consist of four unknown forces, H_a, V_a, H_f, and V_f. Therefore, even though H_a or H_f is removed, the stability of the entire structure as a rigid body is still maintained, and either force can be selected as a redundant force.

Also if the portion of the structure below section *m-n* is considered and the boundary forces acting on the columns indicated as shown in Fig. 1.5*b*, it is apparent that, if H_a is known or removed, the portion *am* will still

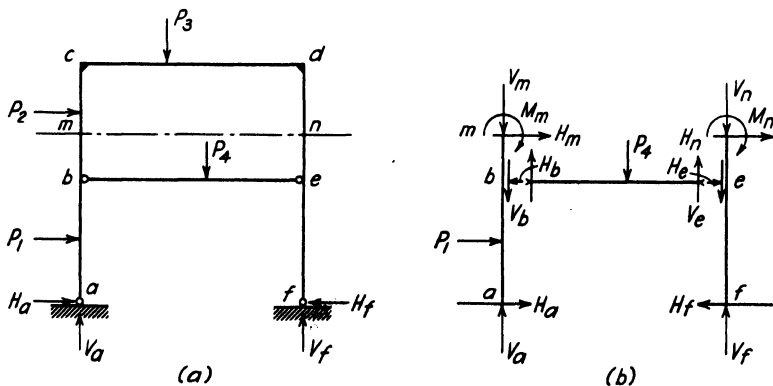



Fig. 1.5

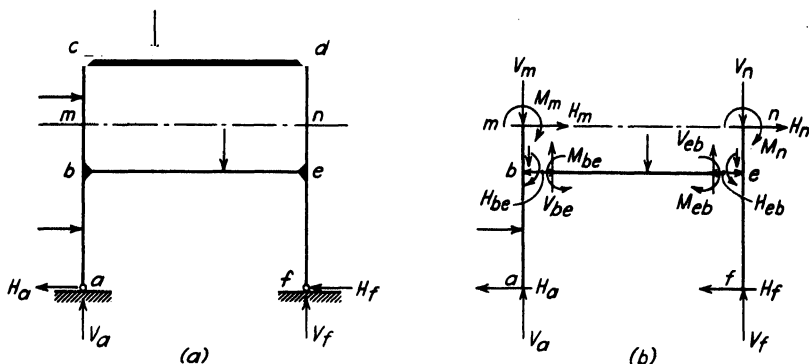



Fig. 1.6

be in equilibrium even if H_b is also removed. Therefore H_b which is equal to H_e is also a redundant and, consequently, the analysis of the structure involves the determination of two redundant forces.

Example 1.3 The frame in Fig. 1.6*a* differs from that in Fig. 1.5*a* only because the member *be* is connected to the columns by a shear and moment connection. The frame is again once redundant externally since either H_a or H_f can be removed. However, if the equilibrium of members *am* and *fn* in Fig. 1.6*b* is considered, it is evident that not only the horizontal force H_{be} can be removed, but also the couples M_{be} and M_{eb}. The structure is therefore four times redundant, for it has one external and three internal redundant forces.

1.5 *Strain Conditions*

Whenever a redundant force is removed from a structure, the elastic curve of the axis of each member changes; that is, the structure deforms to a different position under applied loads. In other words, the specified positions of the boundaries cannot be maintained when the redundant forces are not acting. Thus in Fig. 1.4 if the redundant forces H_a, V_a, and M_a are removed, the axis of the arch will move to some new position as indicated by the broken line in Fig. 1.7*a*. If V_a is now applied with the arch still supported at *b*, Fig. 1.7*b*, another elastic curve is obtained in which the movement of point *a* is indicated by Δ_{ax}', Δ_{ay}', and α_a'. Similarly, for successive applications of H_a and M_a the elastic curves shown in Fig. 1.7*c* and 1.7*d* are obtained.

If the numerical relationships existing between the applied forces and the corresponding horizontal, vertical, and rotational displacements of the section at *a* are established for the four elastic curves shown, it is apparent that the fixed condition in Fig. 1.4 is obtained only if the

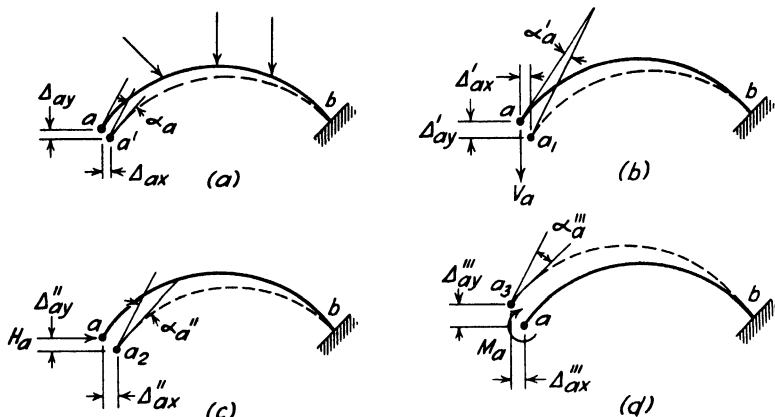



Fig. 1.7

numerical values of H_a, V_a, and M_a satisfy the following strain equations:

$$\begin{aligned} \Delta_{ax}' + \Delta_{ax}'' - \Delta_{ax}''' + \Delta_{ax} &= 0 \\ \Delta_{ay}' + \Delta_{ay}'' - \Delta_{ay}''' + \Delta_{ay} &= 0 \\ -\alpha_a' - \alpha_a'' + \alpha_a''' - \alpha_a &= 0 \end{aligned} \tag{1.3}$$

The various terms in equation 1.3 may be either positive or negative, depending on the assumed direction of the redundant forces. A consistent sign convention for the direction of both forces and displacements must be used.

The corollary statement can be made that in order to keep point *a* from moving, the redundant forces are identical to those that must be applied to restore the position of point *a* if the restraints are removed. This statement is useful in analytical work only if the relationships between forces and displacements are linear or definitely known. This requirement of linearity will be discussed later.

The translation of a point on the axis of a member or at some specific point in the structure, together with the rotation of the cross section about principal axes of inertia, must become familiar concepts since these movements are involved in practically all solutions of structures with redundant forces. In general, the rotation of a cross section is assumed to be identical to the rotation of the tangent to the continuous elastic curve but, as shown later, the two rotations are not identical when the effect of deformations as a result of shear forces is included.

1.6 Summary

The fact that the analysis of statically indeterminate structures must be based on a study of the displacements which the various members can

undergo, both separately and as a unit, has been emphasized in the preceding articles. At this time warning should be given that the assumption of a certain strain condition presumes experimental evidence to justify it. The designer must study carefully the configuration of actual structures and models to secure accurate data as a basis for his calculations. A study of the results of actual measurements as recorded in technical literature is important in estimating the probable accuracy of analytical solutions. Results obtained from mathematical solutions that depend on unverified assumptions must always be regarded critically. Therein lies an important difference between the equations established from equilibrium conditions and those obtained from assumed strain conditions, for the former are unquestionable.

For this reason the methods and procedures explained in subsequent chapters for the analysis of various types of statically indeterminate frame structures give assurance of mathematical accuracy only. The material has been arranged, however, so that the astute observer can incorporate his ideas of the physical action of the structure into the mathematical solution. The design of indeterminate structures requires acumen and sound engineering judgment even more than mathematical ability, but, when carefully conceived, such a structure will often fulfill its function better than a statically determinate one.

Problems

1.1 Classify each structure shown in terms of stability and redundancy. If statically indeterminate, specify the force or forces that you would use as redundants. If the structure is unstable, give reasons why.

1.2 Discuss the strain conditions that might be used for the determination of the redundant forces in structures a, b, g, and h in Problem 1.1.

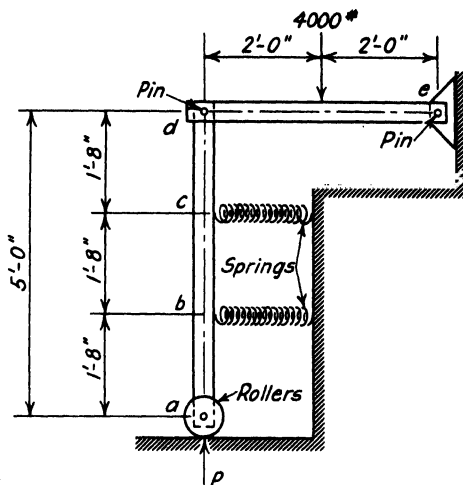
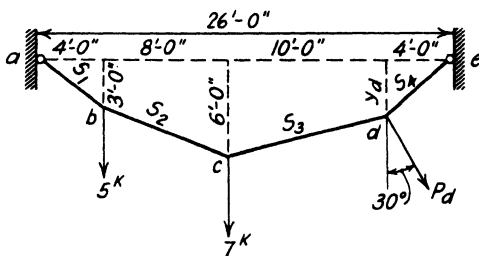
1.3 Construct diagrams for the normal forces, shear force, and bending moments at all sections of beam abc. Draw the diagrams to scale and record the values of the controlling ordinates.

1.4 If the force P acting on the roller support at point a remains vertical (no friction), what is the minimum spring constant that the two identical springs at b and c must have to maintain stability? Consider rigid body movement only and neglect the weight of the members. Note that stability may be defined here as the maintenance of equilibrium for any consistent set of small rigid body displacements.

1.5 A cable with area A and modulus of elasticity E is in equilibrium for the loads and dimensions shown (neglect the weight of the cable).

(a) Determine the magnitudes of P_d, y_d, and all cable stresses S. Use a graphical solution and check algebraically.

(b) Discuss the change in stresses and the position of the cable if all loads are doubled and points a and e remain stationary. How do the numerical values of A and E affect this problem? Can a method of successive approximations be used if it is assumed that points b and c move only vertically?

a
b
c
Fixed
Roller
(a)
a
Hinge
b
c
Rigid joint
(b)
d
Hinge
b
Hinge
c
d
a
Hinge
(c)
b
c
e
L
L
30°
30°
a
d
f
(d)
b
Hinge
Rollers
c
a
Hinge
(e)
(f)
b
d
f
a
c
e
Roller
(g)
b
c
a
d
(h)
Problem 1.1
45°
1414#
4'-0"
6'-0"
2'-0"
1400 lb/ft
2'-0"
a
4'-0"
b
Weight of beam = 150 lb/ft
c
2'-0"
30°
8'-0"
20'-0"
Problem 1.3

Problem 1.4

Problem 1.5

References

1 Hardy Cross, "The Relation of Analysis to Structural Design," *Trans. Am. Soc. C. E.*, Vol. 101 (1936).

2 H. M. Westergaard, "One Hundred Fifty Years' Advance in Structural Analysis," *Trans. Am. Soc. C. E.*, Vol. 94 (1930).

3 Hardy Cross, "The Relation of Structural Mechanics to Structural Engineering," *Publ. Fifth Int. Cong. for Applied Mech.*, 1938.

4 J. I. Parcel and R. B. B. Moorman, *Analysis of Statically Indeterminate Structures*, Chapter I, John Wiley and Sons.

5 H. Sutherland and H. S. Bowman, *Structural Theory*, Chapter 1, John Wiley and Sons.

6 C. H. Norris and J. B. Wilbur, *Elementary Structural Analysis*, pp. 1–36, McGraw-Hill Book Co.

2

Fundamental Principles of External Work and Elastic Strain Energy

2.1 Hooke's Law: Principle of Superposition

The derivation of most fundamental formulas in mechanics of materials begins with a statement that the unit strain is assumed to be proportional to the unit stress or displacements proportional to the load. This linear relationship obviously requires an elastic material, and the deformation of such material produces elastic displacements. Since this assumption, although seldom exact, is the starting point for the mathematical solution of many problems, it seems desirable to repeat the fact that this stress-strain relation, known as Hooke's law, plays an important role in all subsequent analyses. A study of the stress-strain diagrams for various materials under different loading conditions (that are given in many textbooks on strength of materials) will indicate the accuracy of this assumption.

The principle of superposition is useful in structural analysis, for it frequently provides simplification of the mathematical work. Primarily this principle states that the stresses and deformations produced by any number of forces can be obtained by adding the effects of separate equilibrated force systems and that these forces can be considered in any order. The correct use of this principle requires that two conditions be satisfied: first, that the deformations resulting from the various loads be computed with the same physical constants, or that Hooke's law holds;

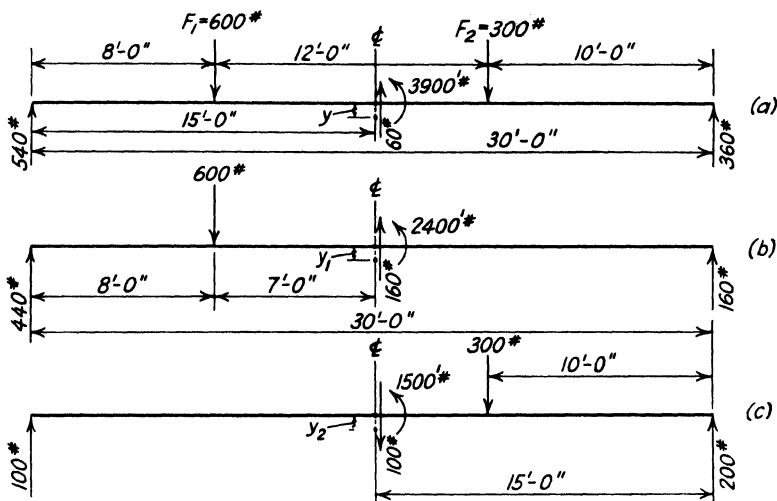



Fig. 2.1

and second, that the deformations resulting from one force system do not affect the deformations caused by another. If either of these conditions is violated, the order in which the loads are considered will affect the final result.

In Fig. 2.1*a* the forces F_1 and F_2 which are applied simultaneously to the beam cause a shear of -60 lb, a bending moment of 3900 ft-lb, and a displacement y at the center of the span. When the forces F_1 and F_2 are applied separately, as shown in Figs. 2.1*b* and 2.1*c*, the combined values of the shears, the bending moments and the displacements are equal to the corresponding values in Fig. 2.1*a*. For these force systems the internal forces are independent of the displacement y.

That this independent relationship does not always exist is readily seen from a study of the internal force S in the members *ac* and *bc*, Fig. 2.2*a*, because of the weight W. The force polygon in Fig. 2.2*b* indicates that the value of S varies with the angle ϕ, which as the members elongate decreases to some value ϕ'. This decrease in the angle also decreases the internal force S, and therefore the relation between W and S is nonlinear as shown in Fig. 2.2*c*. In some structures, such as arches, suspension bridges, and columns subjected to transverse loads, the effect of the movement of the structure on the internal forces may be important and the principle of superposition will not apply directly. Ordinarily, the effect of the distortion of the structure is small and may be neglected in most practical problems, but the engineer should be aware of the effects of deflection, particularly for compression members. In the following

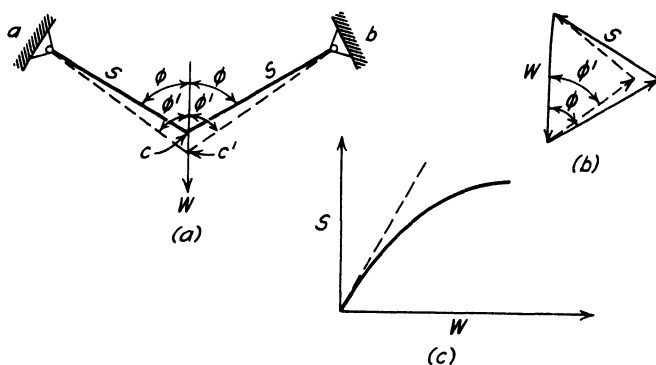



Fig. 2.2

derivations the principle of superposition is assumed to apply unless stated otherwise.

2.2 *External Work*

Any discussion of the external work done by boundary forces acting upon structures must include both the character of the forces and their corresponding displacements, that is, the displacements in the direction of the forces. In the analysis of elastic structures two types of displacements are considered: (1) rigid body displacements, and (2) elastic displacements produced by the deformation of the material. Since the work done by any force F acting through its corresponding displacement Δ is equal to the area under the force-displacement diagram, the student should be able to construct the diagrams and to think in terms of the force-displacement-work relationships that these diagrams represent.

The diagrams in Figs. 2.3*a*, *b*, and *c* represent three commonly assumed conditions that are frequently encountered in the analysis of elastic structures, which may be described as follows.

1. Force is held constant while displacement varies (Fig. 2.3*a*).
2. Force and displacement vary linearly (Fig. 2.3*b*).

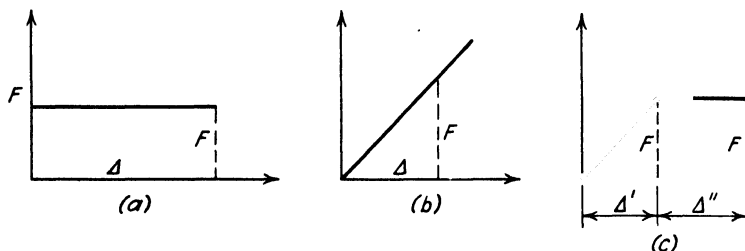



Fig. 2.3

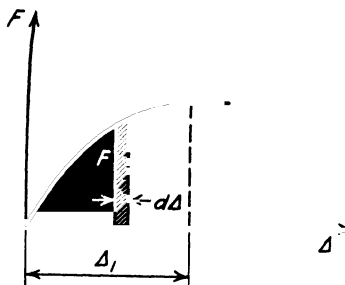



Fig. 2.4

3. Force and displacement increase linearly for a given range and then the displacement increases with constant force (Fig. 2.3*c*).

The areas under the curves which represent the external work W_e are therefore

1. For Fig. 2.3*a*,	$W_e = F\,\Delta$	(2.1*a*)
2. For Fig. 2.3*b*,	$W_e = \frac{1}{2}F\,\Delta$	(2.1*b*)
3. For Fig. 2.3*c*,	$W_e = \frac{1}{2}F\,\Delta' + F\,\Delta''$	(2.1*c*)

In general, the relationship between forces and displacements must be established by the conditions of the problem. For force-displacement conditions that are nonlinear, as in Fig. 2.4, the area under the curve must be obtained either by means of a planimeter or from the integral

$$W_e = \int_0^{\Delta_1} F\,d\Delta \tag{2.2}$$

which can be evaluated only when the algebraic relationship between F and Δ is known. The solution of structural problems involving nonlinear variations of forces and displacements is not easily solved by the use of external work. The reader should study the paper by H. M. Westergaard on the use of complementary energy (Ref. 11).

2.3 *Rigid Body Displacements*

When a structure is moved through a small distance, either translation or rotation, without any change in its shape or size, these movements are classified as rigid body displacements. These displacements are assumed to occur without deformation of the material. For example, if the beam AB (Fig. 2.5) is hinged at A and on rollers at B, it can be rotated about A without disturbing the magnitude of the force system shown. Thus, if the rotation of AB is θ, all forces will change their direction by an amount θ, but their magnitudes will not be modified. In other words, the

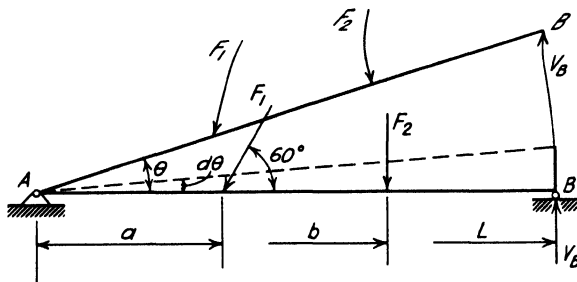



Fig. 2.5

magnitude of boundary forces is not influenced by rigid body motion provided they always maintain the same position relative to themselves and to the structure. This statement is valid whether the boundary forces are statically determinate or indeterminate.

If the beam in Fig. 2.5 is rotated about point A through some infinitesimal angle $d\theta$, the change in the direction of the forces can be ignored and the original distances may be considered unchanged. The external work dW_e is then equal to

$$dW_e = V_B L\, d\theta - F_1 \sin 60°\, a\, d\theta - F_2 b\, d\theta$$

or

$$dW_e = (V_B L - F_1 \sin 60°\, a - F_2 b)\, d\theta \tag{2.3}$$

The terms in parentheses on the right side of equation 2.3 are readily identified as the summation of the moments of all external forces about point A which must equal zero if the beam AB is in statical equilibrium. Therefore, since $d\theta$ is not equal to zero, the external work dW_e must be zero, a condition that should remind us of a familiar principle of physics that the work done by a machine must be equal to the work done on a machine. In this illustration, since all internal elastic energy is neglected (rigid body motion only) and no frictional resistance is considered, the total external work done by and on the structure must be zero, for only potential energy is involved.

2.4 *Principle of Virtual Displacements*

From the preceding discussion of the external work performed by boundary forces acting through any consistent set of rigid body displacements, an interesting principle of structural mechanics can be derived. This statement, which is commonly called the principle of virtual displacements, can be summarized in the following words. If boundary forces are applied to a structure that is in a condition of static equilibrium, the

total external work done by all forces due to any consistent set of rigid body displacements of the structure is zero. As previously illustrated in Article 2.3, this statement involves only the equilibrium conditions and is not concerned with the elastic distortion of the structure.

Example 2.1 The principle of virtual displacements will be used to obtain the magnitude of the vertical reaction V_b for any position of the unit load in Fig. 2.6*a*. To eliminate the boundary forces at *a* and *d*, the displacements at those points will be prevented, as shown in Fig. 2.6*b*. No boundary forces are involved at *c*, and point *b* is assumed to move a vertical distance Δ_b of one unit (any unit of distance). Obviously, we are assuming an extremely small displacement since the arcs are being replaced by perpendiculars. If the external work done by the boundary forces acting through these small virtual displacements is equated to zero, we obtain the equation

$$-V_b(\Delta_b) + (1 \text{ lb})(\Delta) = 0$$

from which

$$V_b = 1 \text{ lb} \frac{\Delta}{\Delta_b}$$

Since Δ_b is assumed equal to unity, V_b is numerically equal to Δ as shown in Fig. 2.6*b*. Thus, when the unit load is at *c*, the value of V_b is 1.6 lb; and when x is equal to 38, V_b is 0.8 lb. The diagram for Δ is therefore an influence diagram for V_b. To obtain an influence diagram for any boundary force by the principle of vertical displacements, a particular movement of the structure is necessary such that the external work involves the desired force as the only unknown. The applied load is, of course, assumed as a unit load in any desired location as illustrated by Fig. 2.6. The structural problems where a solution by the principle of virtual displacements is advantageous are limited. It should be noted that no virtual forces are directly involved in this discussion.

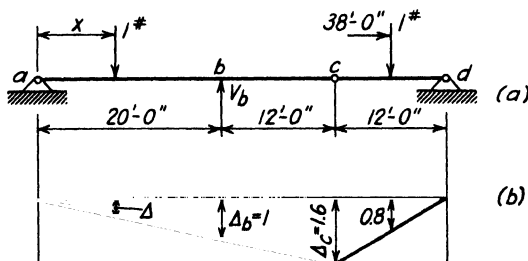



Fig. 2.6

2.5 The Reciprocal Theorem—Betti's Law

In Article 2.3 the external work done by boundary forces acting through linear displacements was discussed and although the illustrations treated positive displacements, that is, in the direction of the forces, it is apparent that equations 2.1*a*, *b*, *c* would hold equally well for negative values of Δ. The external work can therefore be either positive or negative, depending on whether the displacement is in the direction of the force or opposite to it. It will now be proved that a definite relation exists between the forces and linear displacements of two different force systems acting upon a given structure if the material is assumed to be perfectly elastic, that is, the structure will return to its original position when a load is removed.

Let us first consider two force systems that are applied to the beam in Fig. 2.7: the first system consisting of the load P_1 together with the necessary reactions at a and b, and the second system consisting of the load P_2 and reactions. The load P_1 produces the elastic displacements Δ_1' and Δ_2'; the load P_2 produces displacements Δ_1'' and Δ_2''. If P_1 is first applied to the beam, the external work W will be

$$W = \tfrac{1}{2}P_1\,\Delta_1'$$

and if P_2 is now added,

$$W = \tfrac{1}{2}P_1\,\Delta_1' + P_1\,\Delta_1'' + \tfrac{1}{2}P_2\,\Delta_2'' \tag{2.4}$$

Now, if P_1 is removed, the total external work must be equal to

$$W = \tfrac{1}{2}P_1\,\Delta_1' + P_1\,\Delta_1'' + \tfrac{1}{2}P_2\,\Delta_2'' - \tfrac{1}{2}P_1\,\Delta_1' - P_2\,\Delta_2'$$

or

$$W = P_1\,\Delta_1'' + \tfrac{1}{2}P_2\,\Delta_2'' - P_2\,\Delta_2' \tag{2.5}$$

but since only P_2 is now acting on the beam, the total external work must be

$$W = \tfrac{1}{2}P_2\,\Delta_2'' \tag{2.6}$$

It is apparent that equations 2.5 and 2.6 must be equal since they both represent the external work done by P_2. Therefore, in equation 2.5, it is necessary that

$$P_1\,\Delta_1'' - P_2\,\Delta_2' = 0$$

or

$$P_1\,\Delta_1'' = P_2\,\Delta_2' \tag{2.7}$$

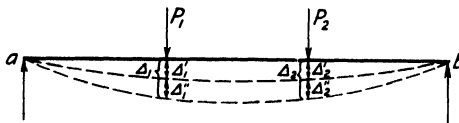



Fig. 2.7

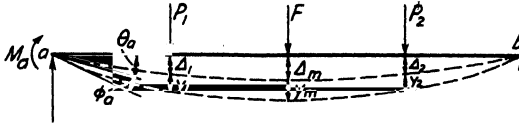



Fig. 2.8

In equation 2.4 the total work done by P_1 and P_2 can now be written in the following form:

$$W = (\tfrac{1}{2}P_1\,\Delta_1' + \tfrac{1}{2}P_1\,\Delta_1'') + (\tfrac{1}{2}P_1\,\Delta_1'' + \tfrac{1}{2}P_2\,\Delta_2'')$$

but since by equation 2.7

$$P_1\,\Delta_1'' = P_2\,\Delta_2'$$

then

$$W = \frac{P_1}{2}(\Delta_1' + \Delta_1'') + \frac{P_2}{2}(\Delta_2' + \Delta_2'')$$

that is,

$$W = \frac{P_1}{2}\Delta_1 + \frac{P_2}{2}\Delta_2 \tag{2.8}$$

The preceding discussion contains two interesting and important facts, namely, (1) the product of P_1 times the corresponding displacement Δ_1'' due to P_2 is equal to P_2 times Δ_2', which is the corresponding displacement due to P_1, and (2) the total external work done by P_1 and P_2 combined is given by equation 2.8 regardless of the order in which they are applied.

The same relations can now be extended by a similar procedure to force systems consisting of several forces. For example, the force system P_1, P_2 in Fig. 2.8 produces the displacements Δ_1, Δ_2, Δ_m, and θ_a, whereas the second system F, M_a causes the displacements y_1, y_2, y_m, and ϕ_a. The work done during the application of P_1 and P_2 has just been shown to be

$$W' = \tfrac{1}{2}P_1\,\Delta_1 + \tfrac{1}{2}P_2\,\Delta_2$$

When the second system F, M_a is applied, the work done by all forces is

$$W'' = P_1y_1 + P_2y_2 + \tfrac{1}{2}Fy_m + \tfrac{1}{2}M_a\phi_a$$

If the first system is removed and the material is assumed to be perfectly elastic, the work performed is

$$W''' = -M_a\theta_a - F\,\Delta_m - \tfrac{1}{2}P_1\,\Delta_1 - \tfrac{1}{2}P_2\,\Delta_2$$

Now, by the principle of superposition, the energy remaining in the beam after the removal of P_1, P_2 should be the same as though only F, M_a had been applied, and therefore

$$W' + W'' + W''' = \tfrac{1}{2}Fy_m + \tfrac{1}{2}M_a\phi_a$$

Substituting the values of W', W'', W''' in the preceding expression gives

$$P_1y_1 + P_2y_2 = F\,\Delta_m + M_a\theta_a \tag{2.9}$$

which proves the following theorem.

If a structure is acted upon by two equilibrated force systems P_1, P_2, $P_3 \ldots$ and F_1, F_2, $F_3 \ldots$ in which the forces P produce displacements Δ in the direction of the F forces, and the F forces cause displacements y in the direction of the P forces, the P forces times the corresponding displacements y will be equal to the F forces times the corresponding displacements Δ.

The reciprocal relation of forces and displacements was stated by Clerk Maxwell in 1864, by Otto Mohr in 1874, and derived in the general form stated here by Betti in 1876.

The reciprocal theorem is an important tool in the analysis of complicated structures by either small-scale models or analytical methods. The following application to the solution of a continuous beam will illustrate its use.

Example 2.2 An influence diagram for the reaction R_a (value of R_a for any value of x) of the continuous beam $ABCD$ (Fig. 2.9*a*) will be obtained by a small-scale model.

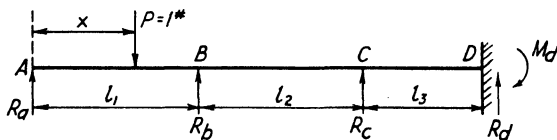



Fig. 2.9a

SOLUTION. Construct a small-scale model as shown in Fig. 2.9*b* in which the span lengths and moments of inertia are proportional to the values in the actual beam. The model should be mounted on ball bearings and supported as shown. If a transverse force F_1 is applied, the axis of the model will take some curve as indicated by the dotted line. The actual values of these displacements can be measured with a microscope or micrometer screw. Now let us regard the unit load P and the accompanying reactions (Fig. 2.9*a*) as one force system, and the loads F_1, F_2, F_3 that

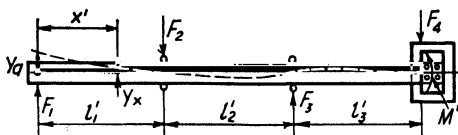



Fig. 2.9b

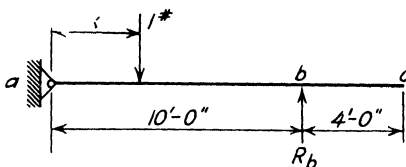
are actually applied to the model (Fig. 2.9*b*) as a second force system. Then, by the reciprocal theorem, equation 2.9, we obtain (if $x'/l' = x/l$)

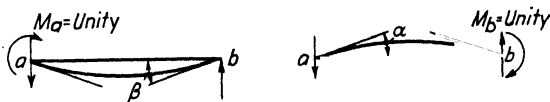
$$-(1\text{ lb})(y_x) + R_a y_a + R_b(0) + R_c(0) + R_d(0) + M_d(0)$$
$$= F_1(0) + F_2(0) + F_3(0) + F_4(0) + M'(0)$$

$$R_a = 1\text{ lb}\frac{y_x}{y_a} \qquad (2.10)$$

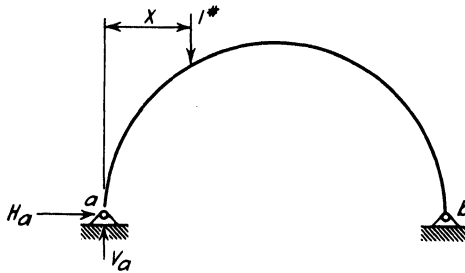
The ordinates to the influence diagram for R_a can therefore be obtained by measuring the displacements y of the elastic curve produced by the F forces. In general, as shown by Müller-Breslau, any influence diagram for a redundant external or internal force can be obtained from some elastic curve, but *not* every elastic curve is an influence diagram. The best test is to apply the reciprocal theorem. The fact that no forces need be measured makes the above procedure a valuable tool for the analysis of extremely complicated structures by means of small-scale flexible models.

Problems

2.1 Construct an influence diagram for the reaction R_b by the principle of virtual displacements.

Problem 2.1

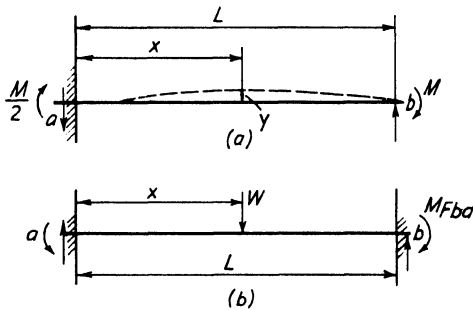
2.2 By means of the reciprocal theorem, show that the angular rotation β at b for a unit moment at a is the same as the rotation α at a for a unit moment at b. Is this statement still valid when the cross section of the beam varies?

Problem 2.2

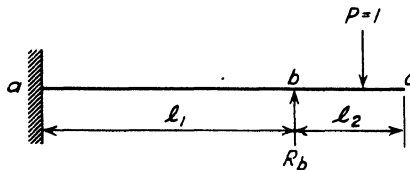
2.3 (*a*) Explain how an influence diagram for H_a for the two-hinged arch shown can be obtained by means of a small-scale model.

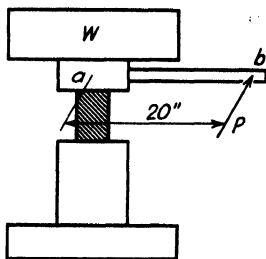
(*b*) Sketch the shape of the diagram from the estimated displacements.

Problem 2.3

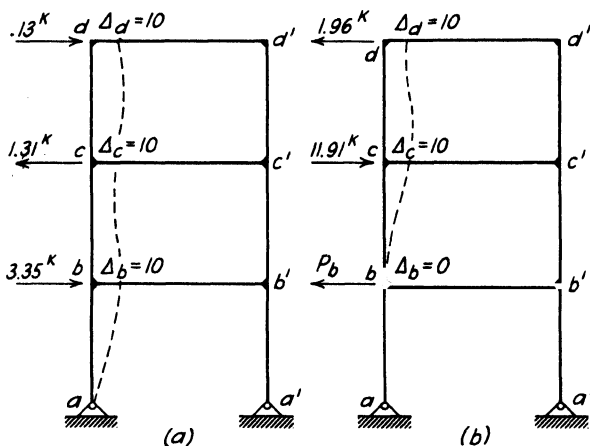
2.4 If the value of y (Fig. a) is given by the equation

$$y = \frac{Mx^2}{4EI}\left(1 - \frac{x}{L}\right)$$

derive an expression for the fixed-end moment M_{Fba} (Fig. b) by means of the reciprocal theorem.

Problem 2.4

2.5 Prove that the influence diagram for R_b for a unit load on the cantilever portion l_2 is linear.

Problem 2.5

2.6 What is the relation between the force P and the load W for the jack-screw shown if the weight W rises $\frac{1}{2}$ in. per revolution of the 20-in. arm ab. Neglect friction and inertia forces.

Problem 2.6

2.7 A monolithic frame structure is subjected to the two force systems and displacements shown in Figs. *a* and *b*. What is the value of P_b in the second force system (Fig. *b*)?

Problem 2.7

2.6 *Elastic Deformation and Strain Energy*

The resultant internal forces acting upon any structural member are usually resolved into normal and transverse components that are parallel and perpendicular, respectively to the axis of the member. The magnitude and distribution of the unit stress and unit strain over the cross section resulting from each component are taken in accordance with the usual assumptions for straight prismatic members composed of a homogeneous elastic material. The deformation and internal work caused by four typical components will now be discussed. These internal forces are: a normal force N (Fig. 2.10) acting at the centroid of the cross section, a bending moment M (Fig. 2.11) about a principal axis through the centroid,

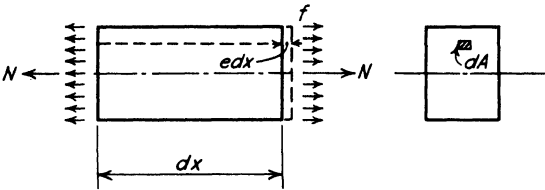



Fig. 2.10

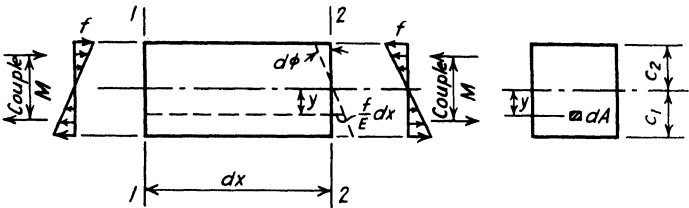



Fig. 2.11

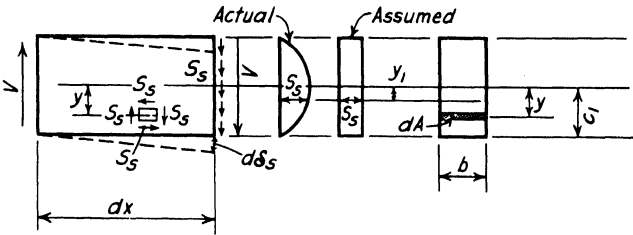



Fig. 2.12

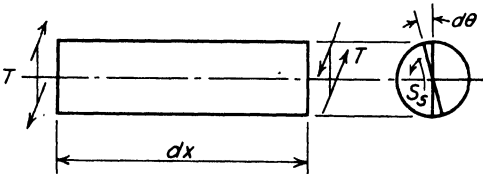



Fig. 2.13

a shear V (Fig. 2.12) parallel to a principal axis, and a torque T (Fig. 2.13) that is calculated with respect to a longitudinal axis passing through the shear center. The shear center will always be on an axis of symmetry and will coincide with the centroid only when both principal axes are axes of symmetry.

The distribution of unit stress and unit strain for a normal component N is assumed to be uniform for any infinitesimal area dA in the cross section of Fig. 2.10. Therefore the value of the unit stress f and the unit strain e is

$$f = \frac{N}{A} \qquad e = \frac{f}{E} = \frac{N}{AE}$$

and the internal work done on any volume of length dx is

$$dW_i = \int \frac{f\,dA}{2}\,e\,dx = \frac{N^2\,dx}{2A^2E}\int dA = \frac{N^2\,dx}{2AE} \tag{2.11}$$

The integration in equation 2.11 is performed over the entire area of the cross section, which will always be the case unless definite limits are indicated. The modulus of elasticity E is assumed constant.

In the foregoing derivation the deformation is assumed to vary linearly with the applied force. It is apparent from equation 2.11 that if the force N is constant over any finite distance L, the total internal work is

$$W_i = \frac{N}{2}\int_0^L e\,dx = \frac{N}{2}\,\Delta L = \frac{N^2L}{2AE} \tag{2.12}$$

In the preceding discussion we have assumed that the stress and the corresponding strain vary together. However, just as for external work, the force can remain constant while the material undergoes strain because of other causes. Thus, if the material undergoes a temperature change or undergoes strain due to other forces, the internal work (assuming N remains constant) is

$$W_i = N\,\Delta L \tag{2.13}$$

where ΔL is the change in length over any distance for which N is constant.

If the change in length is due to a temperature change,

$$\Delta L = \alpha L t \tag{2.14}$$

where α = coefficient of linear expansion
L = length of member
t = change in temperature

When the change in length is caused by another force system,

$$\Delta L = \frac{N'L}{AE} \tag{2.15}$$

where N' is the normal force resulting from any desired load condition. The reader should be able to interpret the internal work in terms of the area under the force-deformation diagram in the same manner as for external work, for the nomenclature in any equation is not adequate to represent all possible conditions. In fact, in the following articles the normal force may be represented by N, S, and u for various types of loads and displacements.

The internal work done by two equal and opposite couples M acting upon an element of length dx (Fig. 2.11) can also be obtained from the integration over the entire cross section of the internal work done by the force acting upon the infinitesimal areas dA and the corresponding displacements.

In the analysis of homogeneous beams subjected to bending moments the unit stress is assumed to vary linearly with respect to y, the perpendicular distance from the neutral axis, in accordance with the equation

$$f = \frac{My}{I}$$

The strain for any distance dx is therefore

$$e\,dx = \frac{f}{E}\,dx = \frac{My}{EI}\,dx$$

The total work done in any element of length dx is therefore

$$dW_i = \int \frac{f\,dA}{2}\,e\,dx = \frac{M^2\,dx}{2EI^2}\int_{-c_1}^{c_2} y^2\,dA = \frac{M^2\,dx}{2EI} \tag{2.16}$$

since

$$\int_{-c_1}^{c_2} y^2\,dA = I$$

Another way of deriving equation 2.16 is to calculate the angular change $d\phi$ between section 1-1 and 2-2, Fig. 2.11, due to the deformation of the material. This angle is equal to

$$d\phi = \frac{e\,dx}{y} = \frac{M\,dx}{EI} \tag{2.17}$$

The internal work is therefore equal to the area under the moment-angle diagram, or

$$dW_i = \frac{M}{2}\,d\phi = \frac{M}{2}\,\frac{M\,dx}{EI} = \frac{M^2\,dx}{2EI} \tag{2.18}$$

As long as the unit stress and unit strain vary linearly, it is apparent that the work done can be expressed in terms of the moment acting upon the element dx and the relative rotation $d\phi$ of the cross sections. If the

moment and rotation vary together, we have the condition expressed by equation 2.18. However, if the moment M is constant during the rotation $d\phi$, the work done is

$$dW_i = M\,d\phi \tag{2.19}$$

Again $d\phi$ can be calculated from equation 2.17 for any type of loading or strain condition.

The internal work done by the unit shearing stress and shearing strain is somewhat more difficult to determine by integration over the cross section because of their nonlinear variation. To simplify the expressions for internal work due to shear a common procedure is first to derive the expressions on the basis of a uniform distribution of unit stress over the cross section and then to multiply by some factor k, which will correct for the actual variation of the shearing stress. Since this actual distribution is dependent on the geometry of the cross section, the factor k can be considered as a shape factor. Thus all rectangular cross sections of a homogeneous beam have a value of k equal to 1.2.

The value of s_s, the unit shearing stress in a homogeneous beam which has no boundary shear, is given by the expression

$$s_s = \frac{V}{Ib}\int_{y_1}^{c_1} y\,dA = \frac{VQ}{Ib} \tag{2.20}$$

However, for the present a uniform distribution of

$$s_s = \frac{V}{A} \tag{2.21}$$

will be assumed.

The transverse displacement $d\delta_s$ over a length dx (Fig. 2.12) for every infinitesimal area dA is then

$$d\delta_s = \frac{s_s}{G}\,dx = \frac{V}{AG}\,dx \tag{2.22}$$

where G is the modulus of elasticity in shear as determined experimentally. The internal work for all infinitesimal areas for a uniform stress distribution is therefore

$$dW_i = \frac{V}{2}\,d\delta_s = \frac{V}{2}\,\frac{V}{AG}\,dx = \frac{V^2\,dx}{2AG} \tag{2.23}$$

However, as the actual stress distribution is not uniform over the cross section, the internal work can be represented in the form

$$dW_i = k\,\frac{V^2\,dx}{2AG} \tag{2.24}$$

The calculation of k for a rectangular section can be found in several textbooks.[1] For an I-section, A should be taken equal to the web area only and k equal to unity.

When the resultant shear V and the displacement $d\delta_s$ do not vary linearly, then equation 2.24 is not applicable. If V remains constant during any relative displacement $d\delta_s$, then the expression for internal work becomes

$$dW_i = V\,d\delta_s \tag{2.25}$$

where $d\delta_s$ can be obtained from equation 2.22 for any desired loading arrangement.

In a similar manner, the internal work done by a torque T acting on an element dx (Fig. 2.13) can be expressed in the form

$$dW_i = \frac{T}{2}\,d\theta \tag{2.26}$$

where $d\theta$ is the angle of twist over the distance dx caused by the torque T. The relation between T and $d\theta$ is not easy to obtain mathematically except for prismatic members with circular cross sections for which

$$d\theta = \frac{T\,dx}{GJ} \tag{2.27}$$

where J is the polar moment of inertia of the cross section.

For noncircular cross sections the value of J must be evaluated for each particular geometrical shape. For example, the twist in rectangular cross sections of width b and depth d can be closely approximated by the equation[2]

$$J = \frac{0.29db^3}{1 + (b/d)^2} \tag{2.28}$$

When the angular rotation $d\theta$ is caused by some other force system, the internal work becomes

$$dW_i = T\,d\theta \tag{2.29}$$

in which $d\theta$ is evaluated by equation 2.27 for any applied torque.

These equations for internal work can also be used to represent elastic strain energy if the original condition of the material is restored when the applied load is removed; that is, equivalent energy is stored in the

[1] See *Theory of Structures* by Timoshenko and Young, p. 219.

[2] Given by Y. W. Tsui in his dissertation on "Stress Distribution in Haunched Polygonal Girder Space Frames," University of Michigan.

material. Therefore the term W_i representing internal work will at times be replaced by the term U which represents the elastic strain energy stored in the material.

Example 2.3. (*a*) The total internal work done in deforming the material of the tension member shown in Fig. 2.14 will be calculated by equation 2.12. Assume E equal to 10^7 lb per square inch.

$$W_i = U = \frac{(10{,}000)^2(20)}{(2)(1)(10^7)} + \frac{(34{,}000)^2(30)}{(2)(4)(10^7)}$$

$$= 533.5 \text{ in.-lb}$$

(*b*) If an additional load P' of 8000 lb is added to the 10,000 lb, what is the internal work done by the original three loads? From equations 2.13 and 2.15,

$$W_i = 10{,}000\,\frac{(8000)(20)}{(1)(10^7)} + 34{,}000\,\frac{(8000)(30)}{(4)(10^7)}$$

$$= 364.0 \text{ in.-lb}$$

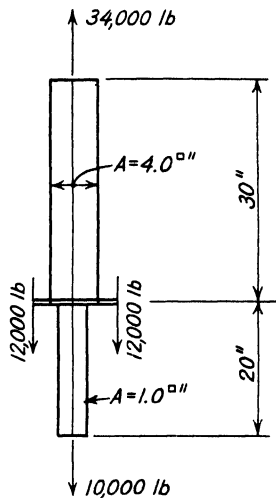



Fig. 2.14

(*c*) The work done by the additional load P' of 8000 lb is

$$W_i = \frac{(8000)^2(20)}{(2)(1)(10^7)} + \frac{(8000)^2(30)}{(2)(4)(10^7)} = 88.0 \text{ in.-lb}$$

(*d*) The total work done by all loads is

$$W_i = 533.5 + 364.0 + 88.0 = 985.5 \text{ in.-lb}$$

which is identical with the internal work obtained if all four loads are applied simultaneously, that is,

$$W_i = \frac{(18{,}000)^2(20)}{(2)(1)(10^7)} + \frac{(42{,}000)^2(30)}{(2)(4)(10^7)} = 985.5 \text{ in.-lb}$$

2.7 *Equality of External and Internal Work*

If the external work is considered simply a change of potential energy with no kinetic energy involved, it is necessary to give the usual, although artificial, restrictions on the manner in which the loads are applied. When

the loads and displacements are varying simultaneously as in Fig. 2.3b, the external work for this linear variation is only one-half the total change in potential energy, and therefore we must assume that some other source of energy is available to assist in applying the weight to the structure at an extremely slow and steady rate, in fact, so slow and steady that no inertial forces are developed—a condition that engineers know, of course, is never actually obtained. When the movements are of such magnitude and character that the inertial forces can no longer be ignored, then the problem involves the principles of dynamics which frequently require a study of potential, kinetic, elastic, and frictional energies that are related by the law of conservation of energy.

When a structure is built of an elastic material and no part or connection develops heat losses through any movement, by the law of conservation of energy the elastic energy stored in the material must be equal to the change in potential energy or external work that can be assigned to the structure. For this condition we can therefore say that the external work is always equal to the corresponding internal work and, since the quantitative determination of the external and internal work has already been discussed, it is now possible to utilize this particular form of the law of conservation of energy.

2.8 *Principle of Virtual Work*

In Article 2.7 the equality of external and internal work was assumed to hold for all consistent sets of forces and elastic displacements that may occur in elastic structures. An important application of this law is the calculation of any elastic displacement by use of an auxiliary or virtual force system. This convenient and general procedure, which is usually called the principle of virtual work, will be explained by several specific illustrations.

The truss in Fig. 2.15*a* is subjected to the external forces F_1, F_2, R_1, and R_2 and the internal forces S which are required for equilibrium conditions. If the external work of this force system is equated to the corresponding internal work, the following expression is obtained:

$$\tfrac{1}{2}F_1\Delta_1 + \tfrac{1}{2}F_2\Delta_2 + R_1 \cdot 0 + R_2 \cdot 0 = \sum \frac{S}{2}\Delta L = \sum \frac{S^2 L}{2AE} \qquad (2.30)$$

in which the summation is made for all members of the truss. However, equation 2.30 contains two unknowns Δ_1 and Δ_2 and therefore does not provide a unique solution. This difficulty can be overcome by assuming that some force P plus the necessary balancing forces for equilibrium are applied to the structure before the desired displacements occur. Thus,

Fig. 2.15

if the displacement Δ_1 in Fig. 2.15*a* is desired, the force system shown in Fig. 2.15*b* is assumed to act before the displacements in Fig. 2.15*a* take place. Consequently, the external work W_e which is due to the unit force in Fig. 2.15*b* acting through the displacements in Fig. 2.15*a* is

$$W_e = (1)(\Delta_1) + (\tfrac{1}{4})(0) + (\tfrac{3}{4})(0) = (1)(\Delta_1)$$

whereas the corresponding internal work W_i for all members is given by

$$W_i = \sum u_1 \, \Delta L = \sum u_1 \frac{SL}{AE}$$

By equating the external work to the corresponding internal work we obtain the value of Δ_1 as

$$\Delta_1 = \sum u_1 \frac{SL}{AE} \tag{2.31}$$

It should be noted that both sides of equation 2.31 have been divided by one unit of force.

After the value of Δ_1 is obtained from equation 2.31, the displacement Δ_2 in Fig. 2.15*a* is most easily calculated by the virtual force system shown in Fig. 2.15*c*. In Fig. 2.15*c* the force system consists of a unit load applied so as to act through the *desired displacement* Δ_2 at L_2 while the balancing forces are placed at points of *known displacement,* in this case, L_0 and L_1. Again we write an expression for the equality of the external and internal work that takes place when the forces in Fig. 2.15*c* act through the displacements in Fig. 2.15*a*, giving

$$(1)(\Delta_2) - 2\Delta_1 + (1)(0) = \sum u_2 \Delta L = \sum u_2 \frac{SL}{AE}$$

from which

$$\Delta_2 = \sum u_2 \frac{SL}{AE} + 2\Delta_1 \tag{2.32}$$

Placing the balancing forces in Fig. 2.15*c* at L_0 and L_1 instead of L_0 and L_4 results in considerably less terms in the summation in equation 2.32 as compared to equation 2.31. In actual trusses which may contain many members the saving in numerical work is considerable.

In applying the principle of virtual work to the calculation of displacements it is apparent from the preceding discussion that the virtual force system should always consist of a force of any magnitude (usually taken as unity) acting through the *desired* displacement and the minimum number of balancing forces which must be applied at points of *known* displacements. The location of the balancing forces should be selected so that the minimum number of terms $u\,\Delta L$ is used in the summation for the internal work. From the preceding discussion it is apparent that the virtual force system can always be *statically determinate* regardless of whether the desired displacements are for determinate or indeterminate force systems.

Example 2.4 The principle of virtual work will be used to determine the vertical displacement of joint L_2 for the truss arrangement and loads as given in Fig. 2.16*a*. The balancing forces of the virtual force system (Fig. 2.16*b*) must be placed at L_0 and L_3 since the vertical displacements of these points are the only known values (zero in this case). Equating

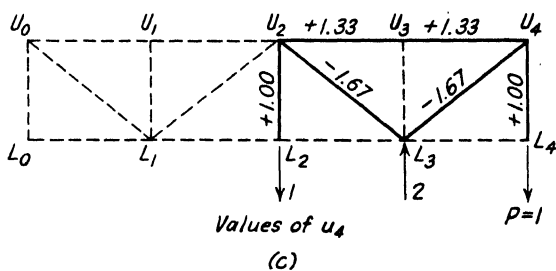
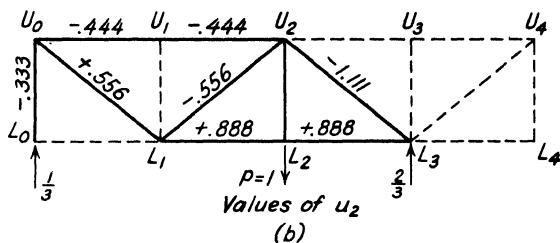
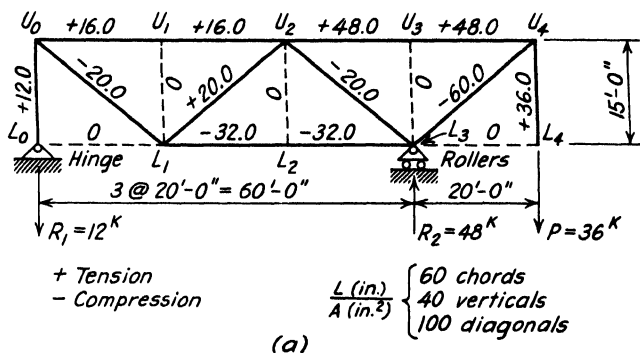



Fig. 2.16

the external work to the internal work as in equation 2.31 and removing the constant term E from the summation gives

$$\Delta_2 = \frac{1}{E} \sum u_2 \left(\frac{SL}{A}\right)$$

The summation of terms on the right side should be obtained by a tabular arrangement as shown in Table 2.1. The vertical displacement of joint

L_2 as determined from Table 2.1 is a minus $4428.2/E$, the minus sign indicating that the movement is upward or opposite to the direction of the unit virtual force.

After the displacement Δ_2 is known, the vertical displacement at L_4 can be determined from the force system in Fig. 2.16*c*, which consists of

Table 2.1

Member	$\frac{L}{A}\left(\frac{1}{\text{in.}}\right)$	S (kips)	$S\frac{L}{A}$ (kips/in.)	u_2 (kips)	$u_2\frac{SL}{A}$
U_0U_1	60	+16	+960	−0.444	−426.2
U_1U_2	60	+16	+960	−0.444	−426.2
U_2U_3	60	+48	+2880	0	0
U_3U_4	60	+48	+2880	0	0
L_0L_1	60	0	0	0	0
L_1L_2	60	−32	−1920	+0.889	−1706.9
L_2L_3	60	−32	−1920	+0.889	−1706.9
L_3L_4	60	0	0	0	0
U_0L_1	100	−20	−2000	+0.556	−1112.0
L_1U_2	100	+20	+2000	−0.556	−1112.0
U_2L_3	100	−20	−2000	−1.111	+2222.0
L_3U_4	100	−60	−6000	0	0
U_0L_0	40	+12	+480	−0.333	−160.0
U_1L_1	40	0	0	0	0
U_2L_2	40	0	0	+1.00	0
U_3L_3	40	0	0	0	0
U_4L_4	40	+36	+1440	0	0
				$\sum u_2\frac{SL}{A} =$	−4428.2

a unit vertical force at L_4 and balancing forces at L_2 and L_3. As previously noted, the balancing forces are placed at points of known displacements. The internal work which is represented by the summation $\Sigma\, u_4 \frac{SL}{A}$ is recorded in Table 2.2. By equating external and internal work we obtain

$$(1)(\Delta_4) + (1)\left(-\frac{4428.2}{E}\right) + (2)(0) = \frac{22453.3}{E}$$

or

$$\Delta_4 = \frac{26881.5}{E} \qquad \text{(downward)}$$

Table 2.2

Member	$\frac{L}{A}\left(\frac{1}{\text{in.}}\right)$	S (kips)	$S\frac{L}{A}$ (kips/in.)	u_4 (kips)	$u_4\frac{SL}{A}$
U_2L_2	40	0	0	+1.00	0.0
U_2L_3	60	+48	+2880	+1.33	+3840.0
U_3U_4	60	+48	+2880	+1.33	+3840.0
U_2L_3	100	−20	−2000	−1.67	+3333.3
L_3U_4	100	−60	−6000	−1.67	+10000.0
U_4L_4	40	+36	+1440	+1.00	+1440.0
				$\sum u_4\frac{SL}{A}$ =	22,543.3

2.9 *Displacements in Beams by Virtual Work*

The principle of virtual work as explained and illustrated in the preceding article is also applicable to the calculation of displacements in beams, frames, and other elastic structures. However, the calculation of the internal work due to bending moments may involve more computations than for the internal work in truss members. A few examples will be presented to illustrate the method and numerical procedure.

The rotation θ_a of the end section of the beam ab in Fig. 2.17a can be calculated by using the virtual force system in Fig. 2.18a which consists of a unit couple applied at a and vertical balancing forces at a and b. The external work W_e due to the virtual forces in Fig. 2.18a acting through the real displacements of Fig. 2.17a is

$$W_e = (1)\theta_a + (2)\left(\frac{1}{L}\right)(0) = (1)(\theta_a)$$

whereas the corresponding internal work W_i due to the bending moments m in Fig. 2.18b acting through the relative rotations $d\phi$ in Fig. 2.17a is

$$W_i = \int_a^b m\, d\phi = \int_a^b m\frac{M\,dx}{EI} = \int_a^b m\, dA$$

By equating W_e and W_i, we obtain

$$\theta_a = \int_a^b m\, dA = \int_a^b \frac{x}{L}\, dA = \frac{A\bar{x}}{L} \qquad (2.33)$$

where A = area of the M/EI diagram

$\bar{x}$ = horizontal distance of the centroid of the area A from end b

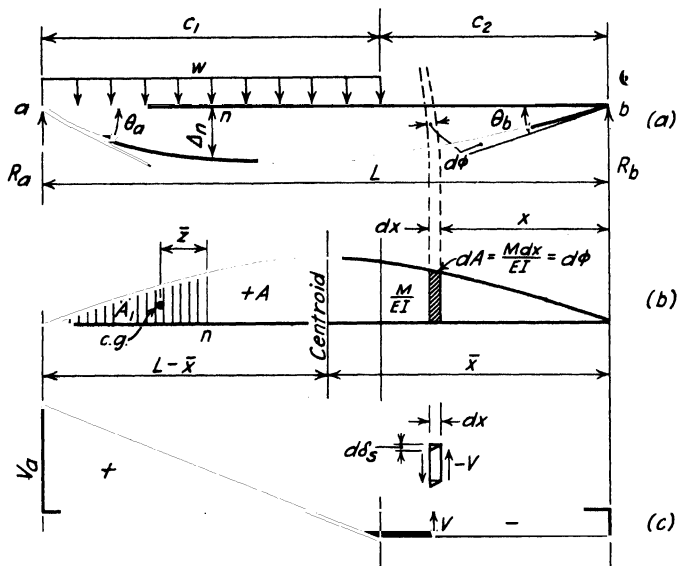



Fig. 2.17

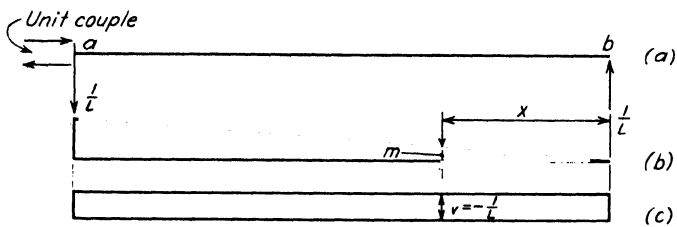



Fig. 2.18

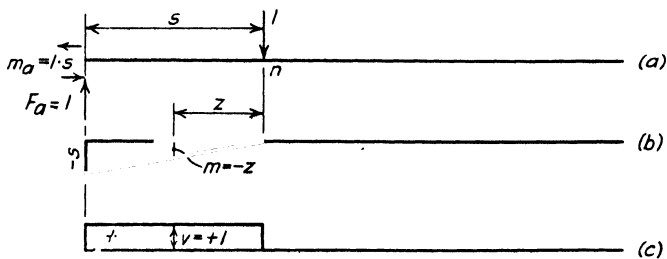



Fig. 2.19

In a similar manner it can readily be shown that the rotation θ_b at support b is

$$\theta_b = \frac{A(L - \bar{x})}{L} \tag{2.34}$$

The displacement Δ_n will now be calculated for the condition that the value of θ_a has already been determined from equation 2.33. The virtual force system in Fig. 2.19*a* consists of a unit load acting through the vertical displacement Δ_n in Fig. 2.17*a* and a balancing force F_a and couple m_a applied at point a where the known displacement and rotation are zero and θ_a respectively. Equating external and internal work due to the virtual forces of Fig. 2.19*b* and the real displacements of Fig. 2.17*a* gives

$$(1)(\Delta_n) - (s)(\theta_a) + (1)(0) = -\int_0^s z\,dA$$

from which

$$\Delta_n = s\theta_a - \int_0^s z\,dA \tag{2.35}$$

or

$$\Delta_n = s\frac{A\bar{x}}{L} - A_1\bar{z} \tag{2.36}$$

since

$$A_1\bar{z} = \int_0^s z\,dA$$

Equations 2.33, 2.34, and 2.36 illustrate the relation between virtual work and the theorems of area moments that will be discussed later.

The displacements in Fig. 2.17*a* produced by the shear deformation can also be obtained by including the internal work due to the virtual shear v acting through the corresponding displacement $d\delta_s$ as given by equations 2.22 and 2.25. By equating external and internal work due to shear only, the reader should be able to obtain the following expressions for θ_a and θ_b from Figs. 2.17*c* and 2.18*c*.

$$\theta_a = \int_a^b v\,d\delta_s = \int_a^b -\frac{kV}{GAL}\,dx = -\frac{kA_v}{GAL} \tag{2.37}$$

where k = shape factor (see Article 2.6)

A_v = area of the shear diagram between a and b

Also, from Figs. 2.17*c* and 2.19*c*,

$$\Delta_n = s\theta_a + \int_n^a v\,d\delta_s = s\theta_a + \int_0^s (1)\left(\frac{kV}{GA}\,dx\right) \tag{2.38}$$

It is apparent from equation 2.37 that θ_a is equal to zero for a simply supported beam without end couples since the total area of the shear diagram is zero. The reader should remember, however, that θ_a is the

rotation of the end cross section at a which, for shear deformations, is not identical to the slope of the elastic curve. The latter is not zero, but is equal to

$$\frac{d\delta_s}{dx} = \frac{kV_a}{GA}$$

Example 2.5 The following numerical values will be assumed for the beam in Fig. 2.17a.

$$c_1 = c_2 = 10' - 0''$$

w, E, I, G, A equal constants in feet and kip units.

From Fig. 2.17a,

$$R_b = 2.5w \qquad M = 2.5wx \qquad \text{for} \quad x \to 0 \text{ to } 10$$

$$M = 2.5wx - \frac{w(x-10)(x-10)}{2} \qquad \text{for} \quad x \to 10 \text{ to } 20$$

or

$$M = 12.5wx - \frac{wx^2}{2} - 50w$$

From Fig. 2.18b,

$$m = \frac{x}{20}$$

and from equation 2.33,

$$\theta_a = \int_0^{10} \left(\frac{x}{20}\right)\left(\frac{2.5wx\,dx}{EI}\right) + \int_{10}^{20} \left(\frac{x}{20}\right)\left(\frac{12.5wx - (wx^2/2) - 50w}{EI}\,dx\right)$$

from which

$$\theta_a = \frac{187.5w}{EI} \quad \text{radians}$$

The displacement Δ at the center of the span is obtained from equation 2.35 as follows.

$$\Delta = (10)\left(\frac{187.5w}{EI}\right) - \int_0^{10} (z)\left[\frac{2.5w(z+10) - (wz^2/2)}{EI}\right] dz$$

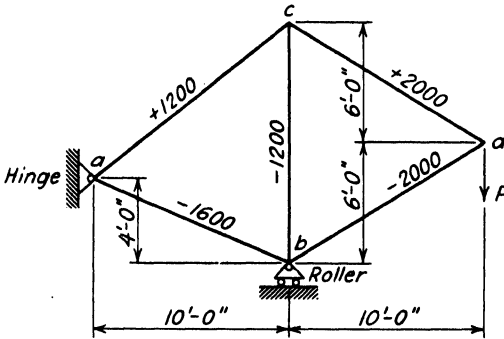
$$= \frac{1875w}{EI} - 833w = \frac{1042w}{EI} \quad \text{ft}$$

The additional displacement at the center of the span due to the shear is (since θ_a due to shear is zero)

$$\Delta_s = \int_0^{10} \frac{kV\,dx}{GA} = \int_0^{10} \frac{k(7.5w - wx)\,dx}{GA} = \frac{k(25w)}{GA}$$

In general, the displacement resulting from shear varies from 3 to 5% of the displacement produced by bending moments for average spans and loads. Heavy concentrated loads on short spans will give larger percentage variations.

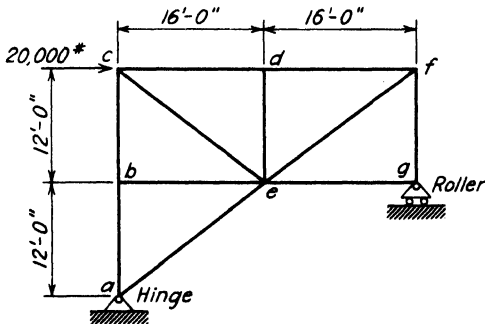
Problems

2.8 The truss shown has SL/A values as recorded on the diagram. Express, in terms of E, the vertical displacement of point d and the horizontal displacement of point c. Use the principle of virtual work (inch and kip units).

Problem 2.8

2.9 From the results in Problem 2.8 calculate the vertical displacement at point d due to a horizontal load of $2P$ at point c.

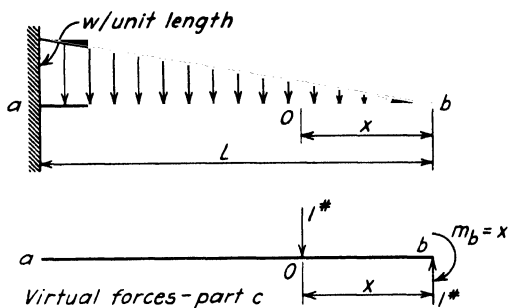
2.10 If the SL/A value for the member ab in Problem 2.8 is changed by a value of plus 400, what will be the corresponding change in the vertical displacement of point d?

2.11 In Example 2.4 the vertical displacement of point L_2 in the truss of Fig. 2.16a was found to be a minus $4428/E$ (upward) and the displacement of point L_0 is zero. Using these two values, determine the vertical displacement of point U_1 by the general theorem of virtual work that is illustrated in Example 2.4. For the first step, draw a diagram showing the virtual force system.

2.12 Determine the vertical and horizontal displacements of point d for the truss shown. Assume that the value of AE for all members is 40×10^6 lb.

Problem 2.12

2.13 A cantilever beam is subjected to a linearly varying load as shown in the diagram. EI is a constant.

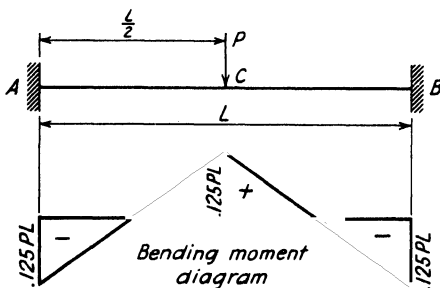
(*a*) Determine the vertical displacement of point b by the principle of virtual work. Express in terms of w, L, and EI.

(*b*) Calculate the rotation of the cross section at b by virtual work.

Problem 2.13

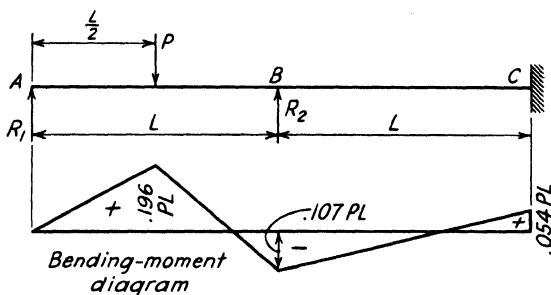
(*c*) From the above results in a and b, determine the vertical displacement at any point O, a distance x from point b by use of the virtual force system shown.

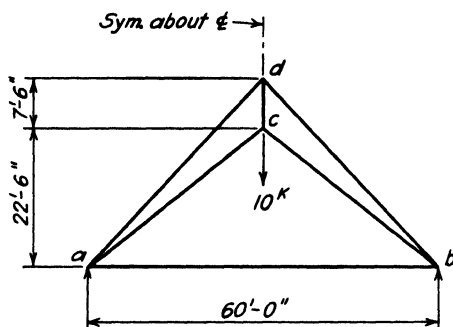
2.14 The bending moment diagram for a fixed-end beam of span L and constant EI with a concentrated load P at the center is shown in the diagram. Calculate the vertical displacement at the center C by the following virtual force systems:

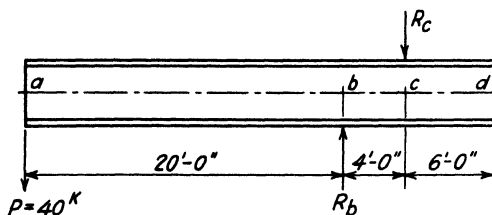
(*a*) A unit vertical load at C and balancing forces at A only.

(*b*) A unit vertical load at C and vertical forces only at A and B.

Problem 2.14

Note that the virtual force system should always be statically determinate. Discuss the analogy between the terms of your equations for external and internal work and the statical moment of the actual bending moment area.

2.15 Calculate the vertical displacement under the load P for the continuous beam ABC. Assume EI is a constant. Use the principle of virtual work and evaluate the integrals from the statical moment of the M/EI diagram.

Problem 2.15

2.16 Calculate the vertical displacement of point c if the stress in the tie ab is 5.72 kips and the area of all members is 4 sq in. Use $E = 10{,}000$ kips per square inch.

Problem 2.16

2.17 What is the vertical displacement of point d in the I beam shown due to shear stresses only if there is no vertical movement of points b and c? Express results in terms of the area of the web A_w and G, the modulus for shear. Neglect the weight of the beam and use a value of k of unity.

Problem 2.17

2.10 Castigliano's Theorem

A general method for computing displacements is given by an important relation between forces and elastic strain energy that is known as

Castigliano's theorem. To explain the meaning of this theorem, let us first consider the loads and displacements of the beam in Fig. 2.20. If the loads P_2 and P_3 are applied to the beam first, the elastic curve will take some position as shown by the ordinates y. When the load P_1 is placed on the beam, the elastic curve will undergo additional displacements as shown by the ordinates z. If the load P_1 is now increased by some increment dP_1, the displacements z_1, z_2, z_3 will be changed by the amounts dz_1, dz_2, dz_3.

The change in the strain energy dU resulting from the addition of the increment dP_1 only is

$$dU = \left(P_1 + \frac{dP_1}{2}\right) dz_1 + P_2\, dz_2 + P_3\, dz_3$$

If Hooke's law holds,

$$dz_1 = \frac{z_1}{P_1}\, dP_1 \qquad dz_2 = \frac{z_2}{P_1}\, dP_1 \qquad dz_3 = \frac{z_3}{P_1}\, dP_1$$

or

$$dU = \tfrac{1}{2} dP_1\, dz_1 + \frac{1}{P_1}(P_1 z_1 + P_2 z_2 + P_3 z_3)\, dP_1$$

However, by the reciprocal theorem, Article 2.5

$$P_2 z_2 + P_3 z_3 = P_1 y_1$$

or, neglecting the term $\frac{1}{2}\, dP_1\, dz_1$,

$$dU = \left(\frac{P_1 z_1 + P_1 y_1}{P_1}\right) dP_1 = \Delta_1\, dP_1$$

which since only P_1 is changed can be written in the form

$$\frac{\partial U}{\partial P_1} = \Delta_1 \tag{2.39}$$

where Δ_1 is the total displacement in the direction of P_1.

Castigliano's theorem, as represented by equation 2.39, can be stated as on p. 43.

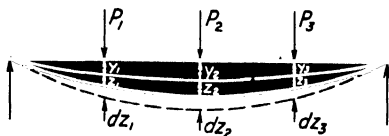



Fig. 2.20

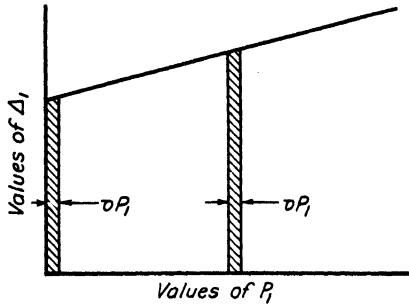



Fig. 2.21

When a structure is acted upon by an equilibrated force system which produces a total internal strain energy U, the partial derivative of U with respect to any force gives the displacement in the direction of that force.

A more direct interpretation of equation 2.39 is obtained from Fig. 2.21 in which the displacement Δ_1, due to all forces on the structure, varies linearly as only the force P_1 is increased. The intercept on the Y axis gives the value of Δ_1 when P_1 is equal to zero. If the force P_1 is increased from any value by an amount ∂P_1, the shaded area, which represents the external work done, must be equal to the change in strain energy ∂U in the structure. That is,

$$\Delta_1 \, \partial P_1 = \partial U$$

or

$$\frac{\partial U}{\partial P_1} = \Delta_1 \tag{2.39}$$

Obviously, as this relation holds for any value of P_1, it can be used when P_1 is equal to zero; that is, the increment of load can start from zero. Consequently, if the displacement is desired at a point where no load is applied, a force P_1 must be assumed in the direction of the displacement and its magnitude reduced to zero after the algebraic expression for the strain energy has been differentiated. This operation implies that, although P_1 may be set equal to zero, $\partial U/\partial P_1$ need not be zero.

If a couple whose moment M is equal to Pc (Fig. 2.22) is applied, the work done by any increase ∂P in both forces P is

$$\partial U = (\Delta' + \Delta'')\, \partial P = \frac{(\Delta' + \Delta'')}{c} \cdot c \cdot \partial P$$

But $(\Delta' + \Delta'')/c$ is equal to the rotation θ, and $c \cdot \partial P$ is the increase ∂M of the applied couple. Therefore any rotation θ can be calculated from

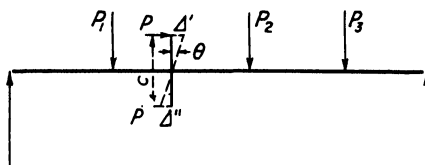



Fig. 2.22

the relation

$$\frac{\partial U}{\partial M} = \theta \tag{2.40}$$

In the preceding discussion dP_1 was replaced by ∂P_1 to indicate that other applied forces (not including reactions) are kept constant while differentiating. Obviously, the terms P_1, P_2, P_3, include the effect of the reactions, for we are treating force systems in equilibrium.

Example 2.6 The application of Castigliano's theorem to the determination of displacements in structures will be exemplified by the calculation of the vertical displacement Δ_B of point B and the horizontal movement Δ_C of point C for the semicircular arch in Fig. 2.23*a*. The internal forces acting upon any right section of the arch at an angle θ with the horizontal are shown in Fig. 2.23*b*. The magnitudes of these forces are

$$N = \frac{P}{2}\cos\theta \qquad V = \frac{P}{2}\sin\theta \qquad M = \frac{Pr}{2}(1 - \cos\theta)$$

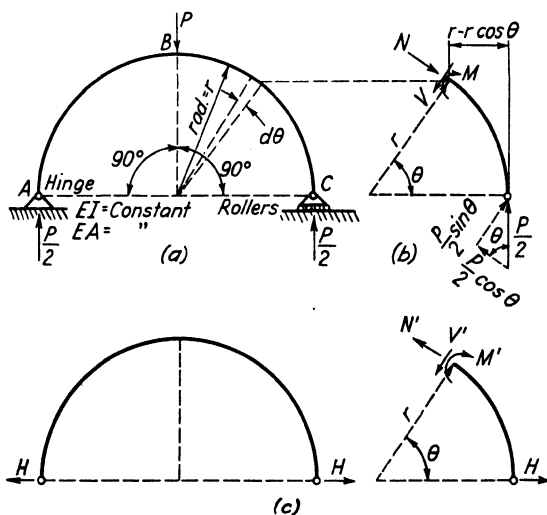



Fig. 2.23

The expression for the total strain energy U in the structure will therefore be (note symmetry)

$$U = \int_A^C \frac{M^2\,ds}{2EI} + \int_A^C \frac{V^2\,ds}{2AG} + \int_A^C \frac{N^2\,ds}{2AE}$$

$$U = 2\int_0^{\pi/2} \frac{[(Pr/2)(1 - \cos\theta)]^2 r\,d\theta}{2EI}$$

$$+ 2\int_0^{\pi/2} \frac{[(P/2)\sin\theta]^2 r\,d\theta}{2AG} + 2\int_0^{\pi/2} \frac{[(P/2)\cos\theta]^2 r\,d\theta}{2AE}$$

since $dU/dP = \Delta_B$,

$$\Delta_B = \frac{1}{EI}\int_0^{\pi/2} \frac{P[r(1 - \cos\theta)]^2 r\,d\theta}{2}$$

$$+ \frac{1}{AG}\int_0^{\pi/2} \frac{P\sin^2\theta\,r\,d\theta}{2} + \frac{1}{AE}\int_0^{\pi/2} \frac{P\cos^2\theta\,r\,d\theta}{2}$$

from which

$$\Delta_B = 0.178\,\frac{Pr^3}{EI} + 0.393\,\frac{Pr}{AG} + 0.393\,\frac{Pr}{AE}$$

Let us assume that the arch is composed of an I-section with the following properties:

$$I = 214 \text{ in.}^4$$

$$\text{Total area } A = 8.23 \text{ in.}^2$$

$$\text{Area of web} = 2.7 \text{ in.}^2$$

$$G = 0.4E \qquad r = 10 \text{ ft} = 120 \text{ in.}$$

If the web area is considered as taking all the shear, then the above value of Δ_B becomes

$$\Delta_B = \frac{1437P}{E} + \frac{44}{E}P + \frac{6}{E}P = \frac{1487P}{E}$$

A comparison of the relative magnitudes of the three terms shows that the displacement due to the bending moment is about 97% of the total. Consequently, for many arch structures and for many frames, the effect of the shear and direct stress may be neglected.

To compute the horizontal displacement Δ_C of point C, the auxiliary force system shown in Fig. 2.23c will be added to the actual force system

of Fig. 2.23*a*. The internal forces at any section due to the load H are

$$N' = H \sin \theta \qquad V' = H \cos \theta \qquad M' = Hr \sin \theta$$

When the force systems of Figs. 2.23*a* and 2.23*c* are both applied, the total strain energy U will be

$$U = 2\int_0^{\pi/2} \frac{\{[Pr(1 - \cos \theta)/2] + Hr \sin \theta\}^2 r \, d\theta}{2EI}$$

$$+ 2\int_0^{\pi/2} \frac{[(P/2) \sin \theta + H \cos \theta]^2 r \, d\theta}{2AG}$$

$$+ 2\int_0^{\pi/2} \frac{[(P/2) \cos \theta - H \sin \theta]^2 r \, d\theta}{2AE}$$

$$\frac{\partial U}{\partial H} = \Delta_C = \frac{1}{EI}\int_0^{\pi/2} 2\left[\frac{Pr(1 - \cos \theta)}{2} + Hr \sin \theta\right](r \sin \theta) r \, d\theta$$

$$+ \frac{1}{AG}\int_0^{\pi/2} 2\left(\frac{P}{2} \sin \theta + H \cos \theta\right) \cos \theta \, r \, d\theta$$

$$+ \frac{1}{AE}\int_0^{\pi/2} 2\left(\frac{P}{2} \cos \theta - H \sin \theta\right)(-\sin \theta) r \, d\theta$$

To obtain the correct value of Δ_C, the auxiliary force H must be reduced to zero, giving

$$\Delta_C = \frac{1}{EI}\int_0^{\pi/2} Pr^3(1 - \cos \theta) \sin \theta \, d\theta$$

$$+ \frac{1}{AG}\int_0^{\pi/2} Pr \sin \theta \cos \theta \, d\theta - \frac{1}{AE}\int_0^{\pi/2} Pr \sin \theta \cos \theta \, d\theta$$

from which

$$\Delta_C = \frac{Pr^3}{2EI} + \frac{Pr}{2AG} - \frac{Pr}{2AE}$$

Although the practical applications of equations 2.39 and 2.40 are often laborious, it is nevertheless a useful tool in solving many problems. The example just solved shows that the numerical work is greatly reduced if the differentiation is performed before the integration. The principal advantage of the method is the ease with which the energy for all internal forces, that is, moments, shear, axial stress, and torque, can be incorporated into the equations. The disadvantage lies mainly in the laborious task of solving the equations that are often involved, although, as shown

later, this difficulty is frequently overcome by replacing the integration by summation.

2.11 *Discussion of Elastic Strain Energy*

Although the corresponding elastic displacements of separate force systems can be added algebraically whenever the principle of superposition applies, this addition does not apply to the corresponding strain energy as the latter quantities are not linear with respect to the forces. This non-linear relation is apparent from equations 2.12 and 2.18 which contain N^2 and M^2 respectively. Therefore, in general, if M_1 and M_2 represent the bending moments of two separate force systems, then the total strain energy of the combined force systems is

$$U = \int \frac{(M_1 + M_2)^2\, dx}{2EI} \tag{2.41a}$$

which is not equal to

$$\int \frac{M_1^{\,2}\, dx}{2EI} + \int \frac{M_2^{\,2}\, dx}{2EI} \tag{2.41b}$$

unless

$$\int \frac{M_1 M_2\, dx}{EI} = 0 \tag{2.41c}$$

For most force systems equation 2.41*c* is not satisfied and therefore the elastic strain energy is not ordinarily expressed by equation 2.41*b*. Fortunately, however, there are certain functions that can satisfy equation 2.41*c*, particularly those functions that involve the internal forces in terms of sin $(n\pi x/L)$ and cos $(m\pi x/L)$, where m and n are integers. Thus, if in Fig. 2.24,

$$M_1 = A_1 \sin\frac{\pi x}{L}, \qquad M_2 = A_2 \sin\frac{2\pi x}{L}, \qquad M_3 = A_3 \sin\frac{3\pi x}{L}$$

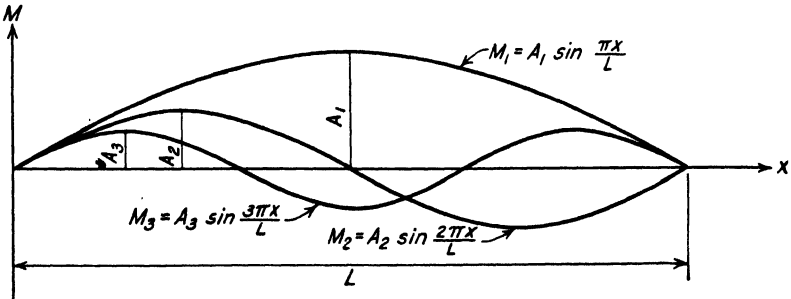



Fig. 2.24

then from inspection of the areas it is apparent that

$$\int_0^L \frac{M_1 M_2\,dx}{EI} = 0$$

if EI is constant.

That this relation also holds for M_1 and M_3 is not apparent from inspection, although it is obviously possible. To prove that equation 2.41*c* is satisfied by the products of $M_1 M_3$, let us write the integral in the following form.

$$\frac{A_1 A_3}{EI} \int_0^L \sin\frac{\pi x}{L} \sin\frac{3\pi x}{L}\,dx$$

$$= \frac{A_1 A_3}{EI} \int_0^L \left[\frac{1}{2}\cos\left(\frac{\pi x}{L} - \frac{3\pi x}{L}\right) - \frac{1}{2}\cos\left(\frac{\pi x}{L} + \frac{3\pi x}{L}\right)\right] dx$$

The total integral is zero since

$$\frac{1}{2}\int_0^L \cos\left(-\frac{2\pi x}{L}\right) dx = 0$$

and

$$\frac{1}{2}\int_0^L \cos\frac{4\pi x}{L}\,dx = 0$$

and therefore

$$\int_0^L \frac{M_1 M_3\,dx}{2EI} = 0$$

Since the preceding discussion is valid for any two unequal values of n, the following conclusion can be made.

If the internal forces or displacements can be represented in terms of $\sin(n\pi x/L)$ and $\cos(m\pi x/L)$, then the elastic strain energy may be expressed by equation 2.41*b* since

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx = 0$$

and

$$\int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\,dx = 0$$

for any two unequal integers n and m.

The use of orthogonal functions for which complete superposition of both energy and displacements is possible is very important in structural problems. Examples of the application of Fourier series, which are the most common type of orthogonal functions, to the solution of structural problems are presented later.

2.12 Calculation of Redundant Forces by Castigliano's Theorem

Any calculation of redundant forces by deriving the necessary strain equations from Castigliano's theorem is based on the assumption that the redundant force can be treated as an applied boundary force. This statement is applicable regardless of whether the redundant forces are defined as external or internal since, in either case, it must be transformed into a boundary force.

The first step in the analysis of a statically indeterminate structure is the selection of the redundant and the reactive forces as discussed in Article 1.4, which should be reviewed by the reader. Since the redundant forces are treated as applied forces, the total strain energy can be expressed in terms of both known and redundant forces. The reactive forces must not appear as unknowns in the equation for the total strain energy. When the bending moments must be expressed by several equations in terms of horizontal and vertical distances x and y, the numerical calculations can be laborious. Several numerical examples will be presented to illustrate the calculation of redundant forces by Castigliano's theorem. It is in such solutions that Castigliano's theorem has particular advantages since no virtual force systems need be introduced. However, when displacements for any known force system are desired, then we have seen that the principle of virtual work is more direct.

Example 2.7 The frame in Fig. 2.25 is once statically indeterminate and the horizontal component H_a will be selected as the redundant force.

Fig. 2.25

(a)

(b)

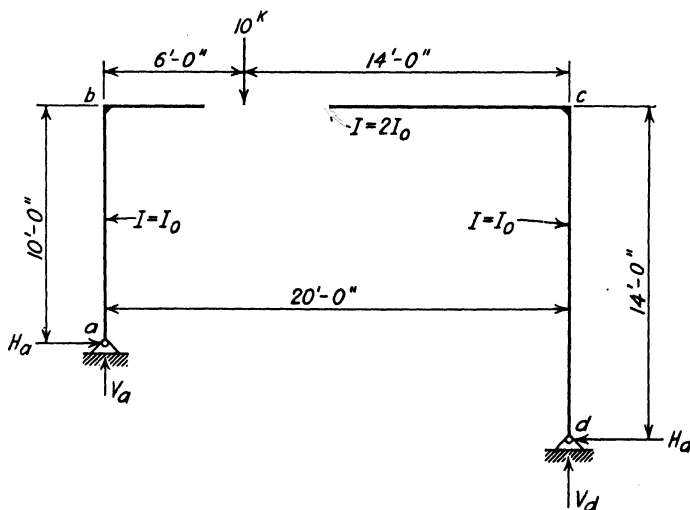
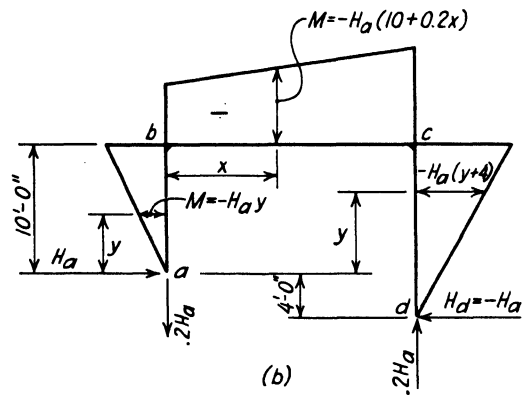
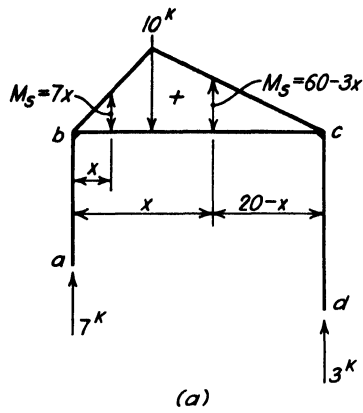



Fig. 2.26

The bending moments at any section of the frame are determined from the sum of the values recorded in Figs. 2.26*a* and *b* which are computed for the 10 kip load and H_a respectively. The total strain energy U due to the bending moments is given by equation 2.41*a* as follows.

$$U = \int_0^{10} \frac{(-H_a y)^2\,dy}{2EI_0} + \int_0^{6} \frac{[7x - H_a(10 + 0.2x)]^2\,dx}{2E(2I_0)}$$

$$+ \int_6^{20} \frac{[60 - 3x - H_a(10 + 0.2x)]^2\,dx}{2E(2I_0)} + \int_{-4}^{10} \frac{[-H_a(y + 4)]^2\,dy}{2EI_0}$$

The derivative of the total elastic strain energy U with respect to H_a gives the displacement of point a in the direction of H_a. Since the support

at a is assumed to be infinitely rigid, no horizontal movement will occur, and the derivative of U with respect to H_a must therefore equal zero. In Example 2.6, the differentiation was performed before the integration, although it is immaterial which order is used. However, in general, a considerable amount of numerical work is saved by differentiating first and then integrating, and therefore this order is used here.

$$\frac{dU}{dH_a} = \frac{1}{EI_0}\left\{\int_0^{10} \frac{2(-H_a y)(-y)\,dy}{2} + \int_0^6 \frac{2[7x - H_a(10 + 0.2x)][-(10 + 0.2x)]\,dx}{(2)(2)} + \int_6^{20} \frac{2[60 - 3x - H_a(10 + 0.2x)][-(10 + 0.2x)]\,dx}{(2)(2)} + \int_{-4}^{10} \frac{2[-H_a(y + 4)][-(y + 4)]\,dy}{2}\right\} = 0$$

Integrating the various terms gives

$$H_a \left|\frac{y^3}{3}\right|_0^{10} + \frac{1}{2}\left| -35x^2 - \frac{1.4x^3}{3} + H_a\left(100x + 2x^2 + \frac{0.04x^3}{3}\right)\right|_0^6 + \frac{1}{2}\left| -600x + 9x^2 + 0.2x^3 + H_a\left(100x + 2x^2 + \frac{0.04x^3}{3}\right)\right|_6^{20} + H_a\left|\frac{y^3}{3} + 4y^2 + 16y\right|_{-4}^{10} = 0$$

from which

$$2722H_a - 2463 = 0$$

or

$$H_a = 0.905 \text{ kips} \quad \text{in the direction assumed}$$

Example 2.8 The truss in Fig. 2.27 is twice redundant internally as can be shown by removing the internal forces in the diagonal members bf and bd. With the redundant forces S_1 and S_2 removed, the stresses S' produced by the two applied loads of 40 kips each have been calculated and recorded on the diagram in Fig. 2.28*a*.

The stresses caused by the redundant force S_1 are conveniently expressed in terms of the coefficients u_1 (Fig. 2.28*b*), which are actually the stresses for S_1 equal to unity. Thus the stress in member bc is $-0.8S_1$ and for any member it is S_1u_1. It should be noted that the member bd on which the unit loads are acting has a u_1 value of unity.

In a similar manner the stresses due to S_2 equal to unity have been calculated and recorded on the diagram in Fig. 2.28*c*. The stress in any member resulting from any value of S_2 is therefore S_2u_2.

Fig. 2.27

(a)

(b)

(c)

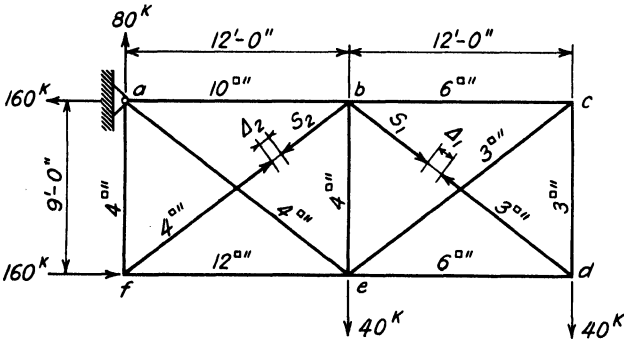
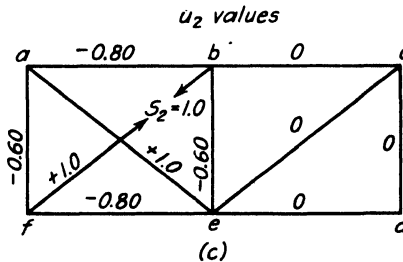
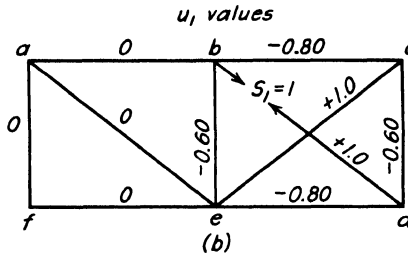
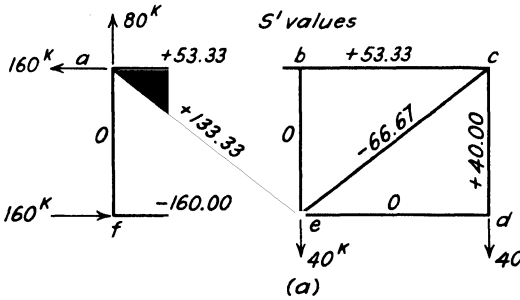



Fig. 2.28

The actual stress in each member of the truss is

$$S = S' + S_1 u_1 + S_2 u_2$$

and the total strain energy U is

$$U = \sum \frac{S^2 L}{2AE} = \frac{1}{E} \sum \frac{(S' + S_1 u_1 + S_2 u_2)^2 L}{2A}$$

Table 2.3

Member	$\frac{L}{A}$	S'	u_1	u_2	$\frac{S'L}{A}u_1$	$\frac{S'L}{A}u_2$	$\frac{u_1^2 L}{A}$	$\frac{u_2^2 L}{A}$	$\frac{u_1 u_2 L}{A}$
ab	1.2	+53.33	0	−0.8	0	−51.2	0	+0.77	0
bc	2.0	+53.33	−0.8	0	−86.1	0	+1.28	0	0
cd	3.0	+40.0	−0.6	0	−72.0	0	+1.08	0	0
de	2.0	0	−0.8	0	0	0	+1.28	0	0
ef	1.0	−160.00	0	−0.8	0	+128.0	0	+0.64	0
ae	3.75	+133.33	0	−1.0	0	+500.0	0	+3.75	0
ce	5.00	−66.67	+1.0	0	−333.3	0	+5.00	0	0
be	2.25	0	−0.6	−0.6	0	0	+0.81	+0.81	+0.81
af	2.25	0	0	−0.6	0	0	0	+0.81	0
bf	3.75	0	0	−1.0	0	0	0	+3.75	0
bd	5.00	0	+1.0	0	0	0	+5.00	0	0
					−491.4	+576.2	14.45	10.53	+0.81

Since the values of Δ_1 and Δ_2 in Fig. 2.27 must be zero for continuous members, by Castigliano's theorem,

$$\frac{\partial U}{\partial S_1} = \Delta_1 = \frac{1}{E} \sum \frac{2(S' + S_1 u_1 + S_2 u_2)(u_1)L}{2A} = 0$$

$$\frac{\partial U}{\partial S_2} = \Delta_2 = \frac{1}{E} \sum \frac{2(S' + S_1 u_1 + S_2 u_2)(u_2)L}{2A} = 0$$

from which the following two strain equations are obtained.

$$S_1 \sum \frac{u_1^2 L}{A} + S_2 \sum \frac{u_1 u_2 L}{A} = -\sum \frac{S' u_1 L}{A}$$

$$S_1 \sum \frac{u_1 u_2 L}{A} + S_2 \sum \frac{u_2^2 L}{A} = -\sum \frac{S' u_2 L}{A}$$

The numerical values of the various summations are recorded in Table 2.3. As the lengths L are in feet and the area A in square inches, the equations have been divided by $12/E$. When the numerical values

in Table 2.3 are substituted in the strain equations, the following solution is obtained.

$$14.45S_1 + 0.81S_2 = 491.4$$

$$0.81S_1 + 10.53S_2 = -576.2$$

from which

$$S_1 = +37.2 \text{ kips}$$

$$S_2 = -57.7 \text{ kips}$$

Example 2.9 The stresses in members bf and bd of the truss in Fig. 2.27 will be calculated for an error in the length of member bd of plus 0.12 in. For this condition, in which the applied loads are removed, the application of Castigliano's theorem gives the following equations:

$$\frac{\partial U}{\partial S_1} = \frac{1}{E}\sum\frac{2(S_1u_1 + S_2u_2)(u_1)L}{2A} = 0.12$$

$$\frac{\partial U}{\partial S_2} = \frac{1}{E}\sum\frac{2(S_1u_1 + S_2u_2)(u_2)L}{2A} = 0$$

By using the values for the summations that are given in Example 2.8 and again dividing by $12/E$ to keep L in inch units, the preceding equations can be written as

$$14.45S_1 + 0.81S_2 = -\frac{0.12E}{12}$$

$$0.81S_1 + 10.53S_2 = 0$$

For a value of E of 10×10^6 lb per square inch, these equations give

$$S_1 = -6950 \text{ lb} \qquad \text{(compression)}$$

$$S_2 = +514 \text{ lb} \qquad \text{(tension)}$$

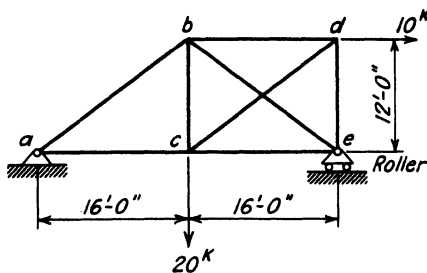
For E equal to 30×10^6, the values of S_1 and S_2 are

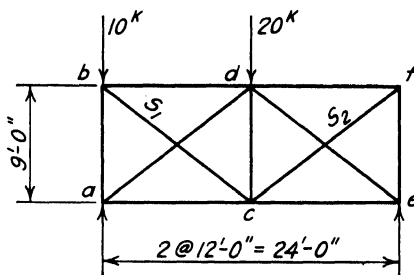
$$S_1 = -20{,}850 \text{ lb}$$

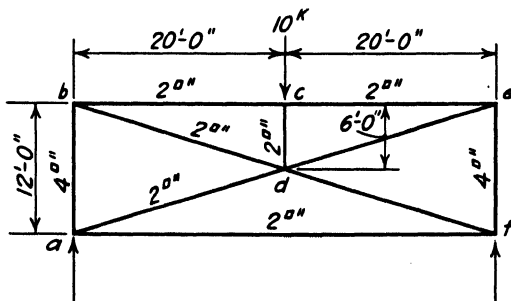
$$S_2 = +1542 \text{ lb}$$

It is apparent that statically indeterminate trusses and frames will probably always have internal stresses due to fabrication and erection and these stresses can be large.

Problems

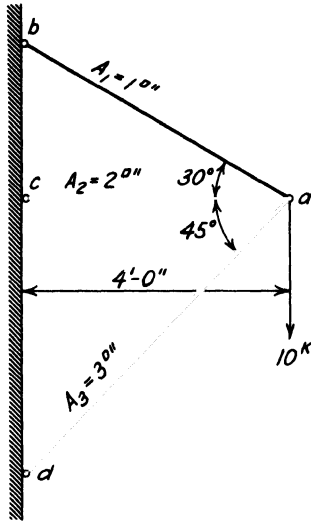
2.18 Calculate the stresses in all members of the truss shown and record the results on a diagram. Assume AE to be constant for all members.

Problem 2.18

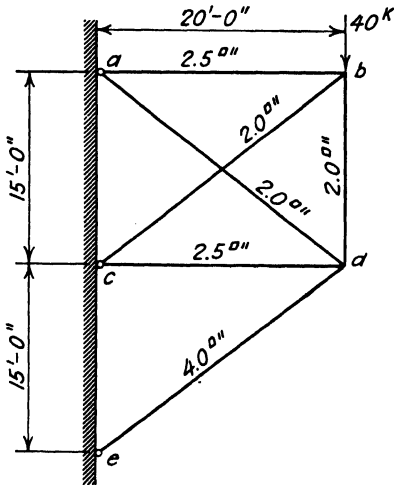
2.19 Determine the value of the stresses S_1 and S_2 by Castigliano's theorem. Assume that the values of AE are constant for all members.

Problem 2.19

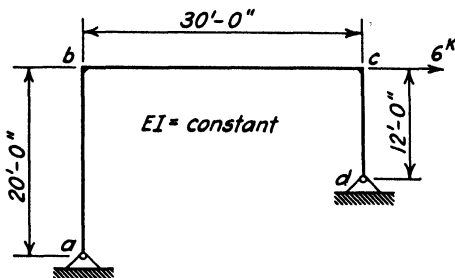
2.20 Calculate the stress in the tie member af using the areas recorded on the diagram.

Problem 2.20

2.21 (*a*) Calculate the stresses for all members for the vertical load of 10 kips. Use constant E value.

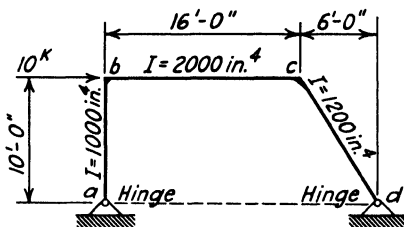
(*b*) Repeat part *a* for a load of 10 kips applied horizontally at point *a* (no vertical load).

(*c*) Calculate the horizontal displacement of point *a* for the vertical load of 10 kips if the value of E is 10×10^3 kips per square inch. Use the principle of virtual work.

Problem 2.21

2.22 Determine the stresses in all members of the structure shown and record values on a diagram. Use the areas given and assume that E is a constant.

Problem 2.22

2.23 Determine the value of the reactions for the rigid frame by Castigliano's theorem. EI is constant.

Problem 2.23

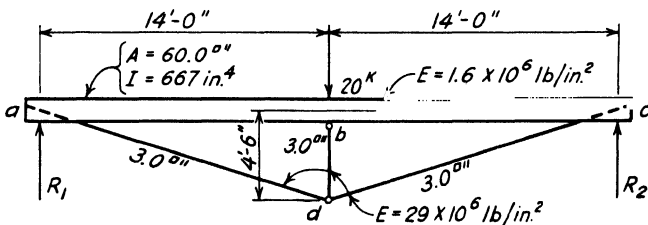
2.24 Calculate the horizontal displacement of point b in the frame of Problem 2.23 by the principle of virtual work. Express in terms of EI.

2.25 (a) Determine the horizontal reaction at a by Castigliano's theorem. Consider the effect of flexure only.

Problem 2.25

(b) Draw the bending-moment diagram and the elastic curve for the frame.

2.26 (a) Determine the reactions at d for the frame in Problem 2.25 the structure is fixed at a and hinged at d.

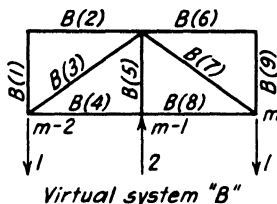
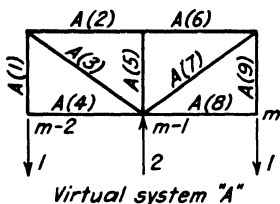
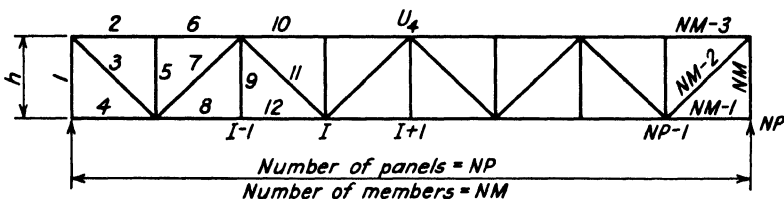
(b) Calculate the vertical displacement of point c in terms of E.

2.27 Calculate the stresses in members bd and ad of the trussed-beam shown. Use center-to-center dimensions.

Problem 2.27

2.28 Construct an influence diagram for the stress in member bd of the trussed beam in Problem 2.27 by use of the reciprocal theorem. (***Hint:*** Consider the beam abc and the member bd forced apart some amount Δ at b

by any two equal and opposite forces. Check your results from the answer in Problem 2.27.)

2.29 If the member *de* in the truss of Problem 2.22 has a temperature rise of 60°F, what stresses will be induced in all members by the increase in length? Use a coefficient of linear expansion of 6.5×10^{-6} and an E value of 30×10^6 lb per square inch.

2.30 Prepare a computer program for the calculation of the vertical deflections of the lower panel points of the truss shown. The truss may have any (up to 100) number of panels but is restricted to constant height and panel length. Truss members are numbered systematically as shown in the sketch.

Problem 2.30

The student should note that the main program can be based on a solution by the principle of virtual work by using alternately the virtual force systems A and B (see sketches). If the deflections at panel points 0 and 1 are first assumed to be zero, the relative deflection at point 2 is

$$y_2 = \sum \frac{SAL}{(\text{area})(E)} + (2)(0) - (1)(0)$$

or the relative deflection y_m at successive panel points is given by the expression

$$y_m = \sum \frac{S[A \text{ or } B]L}{(\text{area})(E)} + 2y_{m-1} - y_{m-2}$$

where m varies from 2 to NP and the summation is made for nine members only.

After the relative deflection Y_{NP} is calculated, the actual deflections Y_{ABS} at any panel point I can be determined from the relative displacements Y_I by the relation

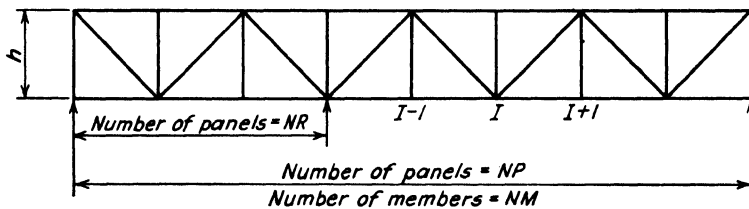
$$Y_{ABS} = Y_I - \frac{IY_{\text{NP}}}{\text{NP}}$$

To check your solution use the following data:

NP = 8 NM = 33

Panel length = 20′-0″ Height = 16′-0″
Cross sectional areas = chords (20.0 sq in.)
Verticals (8.0 sq in.) Diagonals (9.0 sq in.)
Loading 80,000 lb at upper panel point U_4
Prepare data on 36 cards as follows:
Card 1 NM = 33 NP = 8
Card 2 Values of $A(1) \ldots A(9)$
Card 3 Values of $B(1) \ldots B(9)$
Card 4 to 36. (One for each member giving)
Number-length-area-stress

The following final deflections should be obtained for an E value of 30×10^6 lb per square inch (given in inches to three decimals only) 0.000; 0.549; 1.015; 1.382; 1.599; 1.382; 1.015; 0.549; 0.000.

2.31 Prepare a computer program to calculate the ordinates to an influence line for the intermediate reaction of a two-span continuous truss of the type used in Problem 30. The truss may have up to 100 panels with the intermediate support located anywhere (see diagram).

Problem 2.31

Input information will be the same as used in Problem 2.30 plus an integer number NR, which is the number of panels between the left and intermediate support. The student should show that

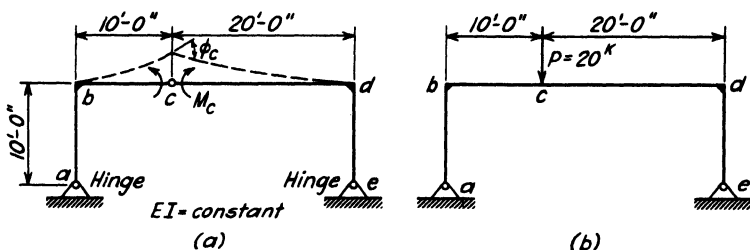
$$R_2 = (1)\left[\frac{Y(I)}{Y_{NR}}\right]$$

For a check on the program the ordinates, to the influence diagram for R_2 with NR is equal to four and with the specified dimensions for Problem 2.30, are as follows:

0; 0.330; 0.611; 0.831; 1.000; 0.831; 0.611; 0.330; 0

2.32 In the rigid frame shown in Fig. *a* members *cb* and *cd* can have a relative rotation ϕ_c at point *c*. However, the two members must have the same displacement at point *c* as indicated. If equal but opposite couples M_c of 1000 ft-lb are applied to members *cb* and *cd* at point *c* determine

(*a*) The angle change ϕ_c at *c* between the members *cb* and *cd* in terms of EI. Use foot units.

(*b*) The vertical displacement of point *c* for the same conditions as in part *a*.

Problem 2.32

(c) The bending moment at point c for the vertical load of 20 kips (Fig. b) if members bc and cd are continuous, that is, bd is a single member. (Use results from parts a and b).

References

1 R. V. Southwell, *Theory of Elasticity*, Oxford Press.

2 S. Timoshenko, *Strength of Materials*, Part I, Chapter X, D. Van Nostrand.

3 Müller-Breslau, "Die neueren Methoden der Festigkeitslehre und der Statik der Baukonstruktionen."

4 Otto Mohr, *Abhandlungen aus dem Gebiete der techischen Mechanik*, W. Ernst and Sohn (1928).

5 E. S. Andrews, "Elastic Stresses in Structures" (translated from Castigliano's "Theoreme de l'equilibre des systems elastiques et ses applications"), Scott, Greenwood and Son, London (1919).

6 J. A. Van Den Broek, *Elastic Energy Method*, John Wiley and Sons.

7 S. T. Carpenter, *Structural Mechanics*, John Wiley and Sons.

8 Carl Shermer, *Fundamentals of Statically Indeterminate Structures*, Ronald Press.

9 A. J. S. Pippard, and J. F. Baker, *The Analysis of Engineering Structures*, Longmans, Green and Company.

10 S. F. Borg, and J. J. Gennaro, *Advanced Structural Analysis*, D. Van Nostrand.

11 H. M. Westergaard "On the Method of Complementary Energy," *Trans. A.S.C.E.*, Vol. 107 (1942).

12 Tung Au, *Elementary Structural Mechanics*, Chapter 8, Prentice-Hall.

3

Geometric Methods for Determining Elastic Displacements

3.1 Summary of Basic Assumptions

The change in shape of elements of infinitesimal length dx due to normal forces N, bending moments M, transverse shears V, and torques T has been discussed in Article 2.6. Although the use of an element of infinitesimal length is necessary in the derivation of formulas by algebraic methods, in graphical solutions finite quantities are usually required. Consequently, a brief review of the deformations of certain elements of finite length in the same plane as the applied normal forces and bending moments is essential before any graphical solutions can be discussed.

If, in equation 3.1, both N, the normal force, and A, the cross-sectional area, are variables, the change in length ΔL for any finite length L is

$$\Delta L = \frac{1}{E}\int_0^L \frac{N}{A}\,dx \qquad (3.1)$$

However, if both N and A are constant, equation 3.1 reduces to the simple form

$$\Delta L = \frac{NL}{AE} \qquad (3.2)$$

If N and A are varying continuously, the exact value of $E(\Delta L)$ from equation 3.1 is represented graphically by the area of the N/A diagram

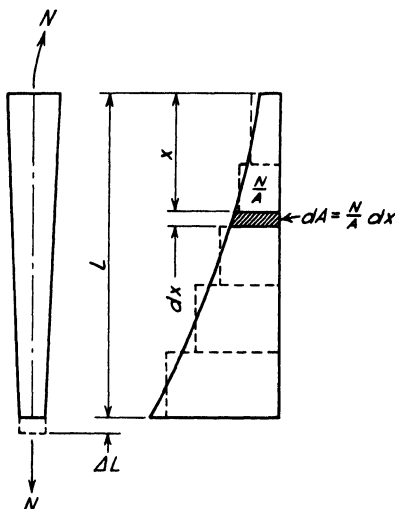



Fig. 3.1

in Fig. 3.1. In other words, the change in length ΔL over any distance L due to the normal forces N can be considered in terms of the area of the N/A diagram divided by E. This area can be determined exactly by integration, or approximately by using Simpson's rule, or even by using a number of rectangles as shown by the dotted lines in Fig. 3.1. The engineer must become accustomed to replacing integration by an approximate summation.

In a similar manner, the strain in elements of finite length Δx resulting from bending moments M about a principal axis of inertia produces a

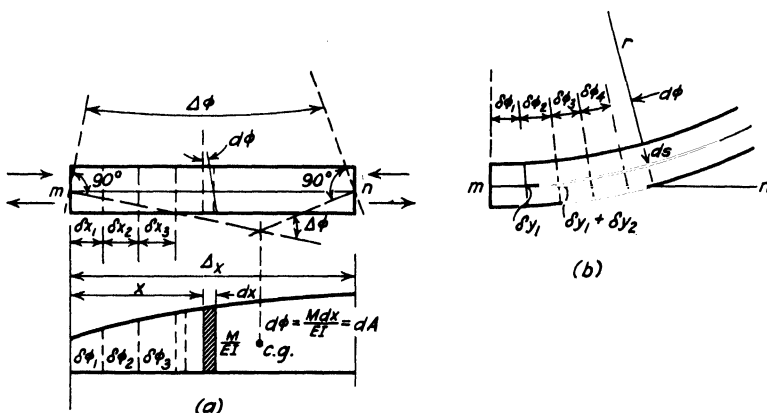



Fig. 3.2

relative rotation $\Delta\phi$ between the two end sections equal to (see equation 2.17)

$$\Delta\phi = \frac{1}{E}\int_0^L \frac{M}{I}\,dx \tag{3.3}$$

Since equation 3.3 is obviously similar in form to equation 3.1, a corresponding geometrical interpretation can be used. Thus $\Delta\phi$ in radians can be considered as the area of M/I diagram in Fig. 3.2*a* divided by E. Again this area may be calculated exactly by integration or approximately by replacing the actual M/I curve by parabolas or straight lines. Such approximations are usually justified as in many practical problems the value of EI is not accurately known.

If the length Δx in Fig. 3.2*a* is divided into smaller elements δx as shown in Fig. 3.2*b*, the relative rotation $\delta\phi$ of the end sections of each element is equal to the corresponding area of the M/I diagram divided by E. Moreover, if each separate element is first deformed and then replaced in position with the other elements (Fig. 3.2*b*), it is apparent that the length Δx of the neutral surface which was originally straight is now curved. Therefore elements of finite length Δx which are treated as composed of many small elements δx will be curved because of the angle changes $\delta\phi$ and the radius of curvature r at any section for originally straight members is

$$r = \frac{ds}{d\phi}$$

However, since ds in practical members can be considered the same as dx,

$$\frac{1}{r} = \frac{d\phi}{dx} = \frac{M}{EI} \tag{3.4}$$

Through reasoning in a similar manner, the reader should prove that the relative transverse displacement between two sections of a straight member due to transverse shears V is equal to the area of the kV/A diagram between the two sections divided by G, the modulus of elasticity in shear. In addition, the relative twist between two sections of a circular member is equal to the corresponding area of the T/J diagram divided by G. Other conditions involving more general states of stress and strain in both straight and curved elements will be presented when necessary.

3.2 *Williot Diagram (Relative Displacement in Trusses)*

In many structures, particularly trusses, the members are subjected to large axial stresses which cause a change in length of the member ΔL in accordance with equations 3.1 or 3.2. If the allowable average unit stress for structural steel is 20,000 lb per square inch, the total deformation

ΔL will ordinarily be about $L/1500$, and consequently it is impossible to show the change in length of the members to the same scale as the truss itself is drawn. To avoid this difficulty a vector diagram that uses only the change in length of the members due to the axial stress was devised by Williot, a French engineer, to show graphically the relative motion of all points in a truss. This information is often used directly in the solution of special problems, such as the calculation of the secondary stresses in large trusses, but primarily, when combined with the Mohr rotation diagram, it provides an extremely practical method for the determination of the actual movement of all joints. The construction of the Williot diagram will therefore be discussed in detail.

If three points *a*, *b*, and *c* are connected by any three members as in Fig. 3.3*a*, and if *ab* and *bc* both shorten 0.2 in. while *ac* elongates 0.3 in., the new position of *c* can be determined if the positions of *a* and *b* are known. For instance, if point *a* remains in position, then point *b* must be somewhere on an arc 1-1 whose radius is (ab — 0.2 in.) (not to scale), whereas *c* will be somewhere on arc 2-2 whose radius is (ac + 0.3 in.). However, before the position of *c* on arc 2-2 can be determined, the position of *b* on arc 1-1 must be known or assumed. A common assumption is to assume that the member *ab* does not rotate and consequently *b* moves to b^1. If member *ab* does actually rotate, the motion of all points due to this rotation must be added vectorially as described later.

Now, if we assume that *b* moves to b^1, then, since the deformation has been shown to be extremely small as compared to the length of the member, the same movements can be obtained with sufficient accuracy by using perpendiculars as shown in Fig. 3.3*b* instead of arcs. Moreover, it is now apparent that the movements represented by these perpendiculars can be drawn separately to a large scale as in Fig. 3.3*c*, which gives the motion of *c* relative to *a* and *b* as a vector quantity.

After the position of point *c* with respect to *a* and *b* has been obtained, the same procedure can be followed to obtain the motion of another point *d* (Fig. 3.4*a*) which is connected to *b* and *c*. The relative moments of all three points *b*, *c*, and *d* are shown in Fig. 3.4*a*. The member *bd* shortens 0.2 in., which moves *d* toward *b* by that amount, whereas the member *cd* elongates 0.15 in., which moves *d* away from *c* by that amount. The intersection of the perpendiculars 5 and 6 will therefore give the movement of *d* with respect to *b*, *c*, and *a*. The construction in Fig. 3.4*b* is so arranged that the displacement vectors need not be duplicated for each joint, but are drawn only once. In this respect, the diagram is similar to a stress diagram for a truss which combines force polygons for each joint without repeating the stress vectors.

Example 3.1 The truss in Fig. 3.5 is supported by a hinge at *a* and by rollers at *h*, which must therefore move parallel to the surface or 45° to

Fig. 3.3

Fig. 3.4

Fig. 3.5

Fig. 3.6 **Graphical solution for displacement of a truss.**

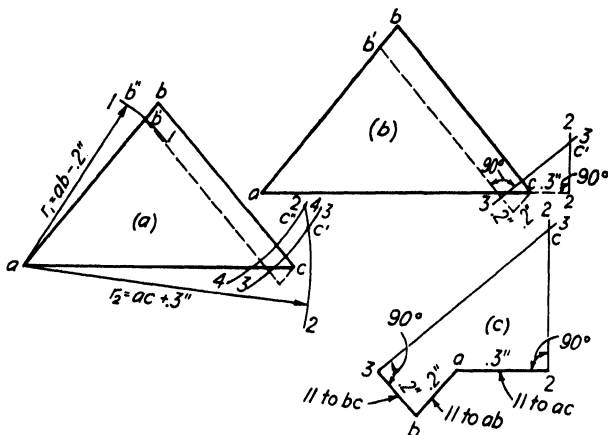
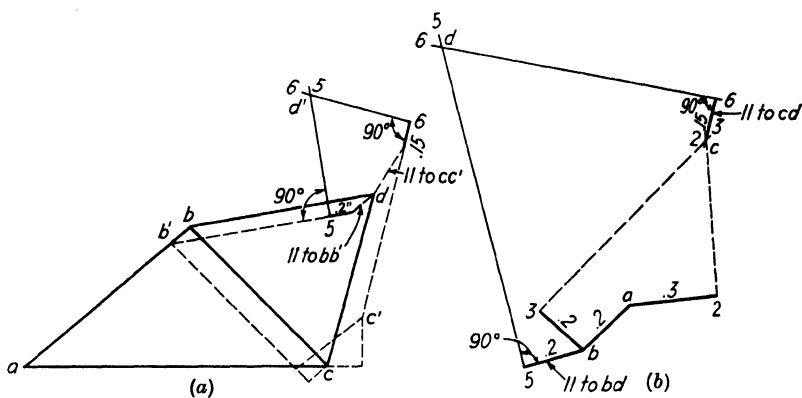
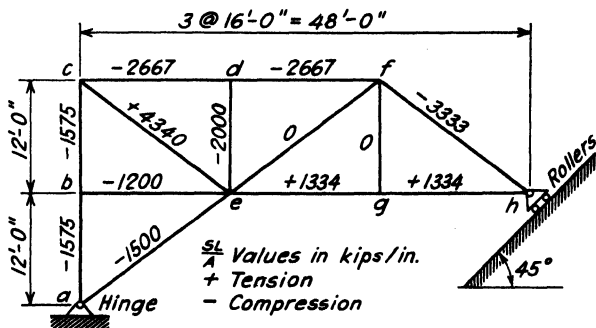
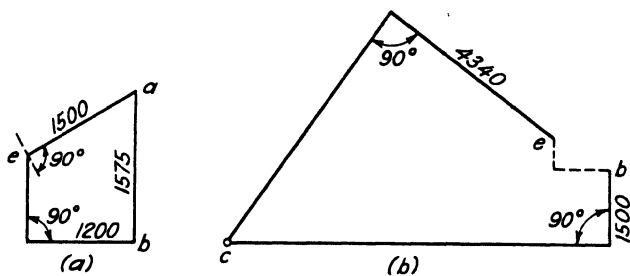



Fig. 3.7

the horizontal. The value of SL/A for each member in kips per inch have been recorded on the diagram of the truss. These values have been used to construct a Williot diagram, which is shown by the lower case points in Fig. 3.6. The graphical construction was started by assuming that point a does not move, which is correct, and that member ab does not rotate, that is, b moves vertically towards a, which we shall determine later to be incorrect. A decimal scale should be used, and the general shape of the vector diagram first sketched out roughly to determine where point a should be placed on the paper.

The first operation is to locate b with respect to a by scaling off 1575 units downward. The direction must be downward as the member is in compression, and therefore b moves toward a. The movement of e with respect to a and b can now be determined by the construction shown in Fig. 3.7a. It so happens that the perpendicular to ae passes through point e, which is a special case. After point e is located, the position of point c is determined from b and e by the construction illustrated in Fig. 3.7b. The locations of the remaining points in the Williot diagram of Fig. 3.6 are made in the following order: d, f, g, and h. The entire graphical construction should be studied until each step is thoroughly understood.

3.3 *Rotation Diagrams (Actual Displacements in Trusses)*

The Williot diagram determines the actual displacement of the joints only if the two reference points from which the construction is started have their correct position with respect to each other. Sometimes this condition can be realized for a particular situation, but more frequently it is necessary to start with a known fixed point and some reference axis that is assumed not to rotate. This assumption was made in constructing the Williot diagram in Fig. 3.6 where a is the fixed point and ab the reference axis. If the deflected position of the truss is drawn by using the relative displacements from the Williot diagram, a diagram like

Fig. 3.8

Fig. 3.8*a* is obtained, in which the displacements are, of course, greatly exaggerated. Since point *h* is now some distance off the support, it is apparent that the original assumption that member *ab* does not rotate is incorrect and that the entire structure must be rotated as a rigid body about point *a* until point *h* comes back to the support. In other words, point *h* can only move in a direction that is 45° to the horizontal. As all movements are small, the rigid body displacement hh^1 in Fig. 3.8*a* can be taken perpendicular to original direction *ah*, and the magnitude of the displacement is the vector hh^1 in Fig. 3.6.

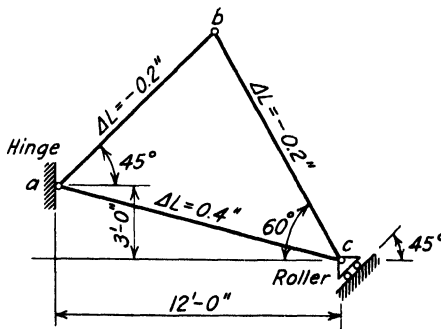
The movements of the other points due to this rotation θ are shown in Fig. 3.8*b*, where it can be seen that each point moves perpendicular to a line connecting it with point *a* and the magnitude of the motion depends upon the distance from *a* and the angle θ. The magnitude of θ is obtained from the vector hh^1 in Fig. 3.6. Professor Mohr was the first one to point out that the rigid body movement of all points produced by the rotation of the structure can be obtained graphically, if one displacement such as hh^1 in Fig. 3.8*b* is known, by drawing a diagram of the truss rotated 90°

as in Fig. 3.8*c*. For example, in Fig. 3.8*b* if the distance *ah* is made to equal hh^1 to some scale, to the same scale *ad* equals dd^1 and *ac* equals cc^1, etc. If this vector diagram is now rotated until *ah* is in the direction of hh^1 (Fig. 3.8*c*) that is rotated 90°, the displacement hh^1 will be given both in magnitude and direction by *HA* and the rotated truss diagram, or Mohr rotation diagram, in Fig. 3.8*c* becomes a vector diagram for all displacements because of rotation.

If the Mohr rotation diagram is drawn to the same scale as the Williot diagram and superimposed upon it so that the fixed points *A* and *a* coincide (see Fig. 3.6), the actual displacement of any point is the vectorial sum of the displacement vectors of the two diagrams. As an illustration, in Fig. 3.6, the rotation vector *Da* for point *d* is added to the relative motion *ad* to give the resultant displacement *Dd*. Similarly, *Bb*, *Cc*, *Ee*, etc., represent the actual displacements of points *b*, *c*, *e*, respectively. It is customary to arrange the two diagrams such that the resultant motion is measured from the point on the rotation diagram to the point on the Williot diagram.

If the Williot diagram is started from an assumed fixed point but, with an actual fixed reference axis, a correction will be necessary only for rigid body translation, but if neither an actual fixed point nor a fixed axis is used, a correction must be made for both translation and rotation. Such corrections are not difficult to execute, and they often provide some interesting graphical solutions. By choosing a reference axis with a small amount of rotation, the rotation diagram will naturally be small, which is of considerable importance in large trusses.

Problems

3.1 The change in length of each member of the truss is recorded on the diagram. Make the following graphical constructions to scale on $8\frac{1}{2} \times 11$ paper.

Problem 3.1

(a) Draw a Williot diagram using a as a fixed point and ab as a fixed direction.

(b) Construct a Mohr rotation diagram to the same scale with respect to point a.

(c) Indicate the vector giving the absolute displacement of point b and record its numerical value.

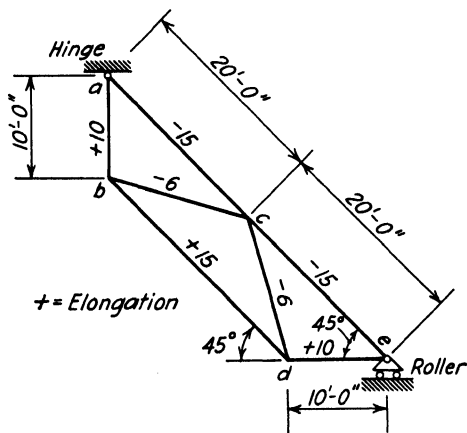
(d) Check the vertical displacement of point b by the principle of virtual work.

3.2 (a) Construct a Williot diagram for the truss in Problem 2.8 to scale with a as a fixed point and ac as a fixed direction.

(b) Draw a Mohr rotation diagram for rotation about point a to the same scale.

(c) Measure and record the vertical displacement of point d.

3.3 Solve Problem 2.12 graphically.

3.4 (a) The truss shown in the diagram is hinged at a and supported on a roller at point e which must therefore move horizontally. The SL/A values are recorded on the diagram. Construct a Williot diagram assuming b as a fixed point and bd as a fixed direction. Show the deformed position of the truss by dotted lines.

Problem 3.4

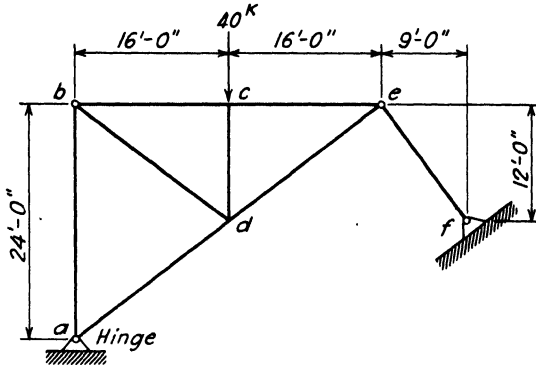
(b) Show that if point a on the deformed truss is moved back to its original position by translation with no rotation, the vector ab in the Williot diagram represents the vectorial motion of all points due to this translation. In other words, the deformed truss moves as a rigid body through the distance and in the direction of the vector ab.

Now prove that the resultant motion of any point such as point c on the deformed structure when point a has been restored to its original position is equal to and in the direction of the vector ac.

(c) Construct a Mohr rotation diagram for rotation about point a.

(d) Measure the horizontal motion of point c and check by the principle of virtual work.

3.5 Determine the displacements of the joints in the truss shown by a graphical solution. Use AE equal to 40,000 kips for all members.

(*Hint*: As f is also a fixed point it will coincide with a in the Williot diagram. Why? The rotation vector for point e when the truss is rotated about point a if added vectorially to the vector ae of the Williot diagram must provide a resultant absolute displacement that will satisfy the movement of point e with respect to point f in the Williot diagram. This condition, of course, means that the absolute motion of point e must have a component in the direction of ef equal to the change in length of ef.)

Problem 3.5

3.4. *Theorems of Area Moments (Relative Displacements in Beams)*

In Article 3.1 the relative rotation $\Delta\phi$ of two normal sections of a straight beam which are a distance Δx apart (Fig. 3.2a) was shown to be numerically equal to the area of the M/EI diagram between the two sections. This relation between the area of the M/EI diagram and the rotation of one cross section (or tangent) with respect to another cross section (or tangent) for any continuous elastic curve is called the first theorem of area moments. It is now customary to refer to the angle $\Delta\phi$ between any two tangents to an elastic curve, such as m and n in Fig. 3.2a, as an angle change. The area of the M/EI diagram gives any angle change in magnitude, direction, and position since the tangents must intersect on a normal line passing through the centroid of the area. For this reason any angle change can be represented by a vector quantity as shown in Fig. 3.9, in which $\Delta\phi_1$, $\Delta\phi_2$, $\Delta\phi_3$ are often designated as elastic weights.

Furthermore, the angle changes or the summations of many angle changes are very small in practical beams, in fact so small that the rotation θ (Fig. 3.9) of any tangent from its original position may be assumed as

$$\theta = \tan\theta = \sin\theta$$

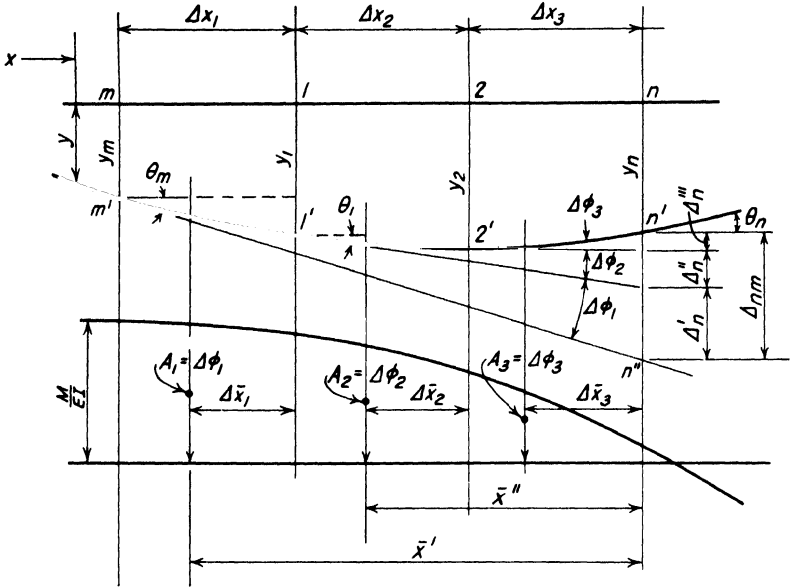



Fig. 3.9

This assumption means that the distance along the arc of the elastic curve in Fig. 3.2*b* can be replaced by the original distances Δx, and the displacements Δy can be measured perpendicular to the original axis *mn*.

When these assumptions are made, the position of any point n' in Fig. 3.9 with respect to any point m' can be determined directly from the slopes or changes in slope of the tangents. From the geometrical relationships shown in Fig. 3.9 and the assumptions previously discussed, the reader should be able to obtain for himself the following equations:

$$\Delta\phi_1 = A_1 \qquad \Delta\phi_2 = A_2 \qquad \Delta\phi_3 = A_3$$

$$\theta_1 = \theta_m - \Delta\phi_1 \qquad \theta_2 = \theta_1 - \Delta\phi_2 \qquad \theta_n = \theta_2 - \Delta\phi_3$$

$$\theta_n = \theta_m - \Delta\phi_1 - \Delta\phi_2 - \Delta\phi_3 \tag{3.5}$$

$$y_1 = y_m + \theta_m\,\Delta x_1 - \Delta\phi_1\,\Delta\bar{x}_1 = y_m + \theta_m(\Delta x_1 - \Delta\bar{x}_1) + (\theta_m - \Delta\phi_1)\,\Delta\bar{x}_1$$

$$y_2 = y_1 + \theta_1\,\Delta x_2 - \Delta\phi_2\,\Delta\bar{x}_2 = y_1 + \theta_1(\Delta x_2 - \Delta\bar{x}_2) + (\theta_1 - \Delta\phi_2)\,\Delta\bar{x}_2$$

$$y_n = y_2 + \theta_2\,\Delta x_3 - \Delta\phi_3\,\Delta\bar{x}_3 = y_2 + \theta_2(\Delta x_3 - \Delta\bar{x}_3) + (\theta_2 - \Delta\phi_3)\,\Delta\bar{x}_3$$

Also

$$y_n = y_m + \theta_m(\Delta x_1 + \Delta x_2 + \Delta x_3) - \Delta\phi_1\bar{x}' - \Delta\phi_2\bar{x}'' - \Delta\phi_3\,\Delta\bar{x}_3$$

The values θ_m and $\tan \theta_m$ have obviously been interchanged at anytime, and $(\theta_m - \Delta\phi_1)$ has been taken equal to $\tan(\theta_m - \Delta\phi_1)$. However, since

$$\tan(\theta_m - \Delta\phi_1) = \frac{\tan \theta_m - \tan \Delta\phi_1}{1 + \tan \theta_m \tan \Delta\phi_1} \tag{3.6}$$

it is apparent that we are neglecting the term $\tan \theta_m$ times $\tan \Delta\phi_1$ in our calculations in addition to the assumption that $\tan \theta$ is numerically equal to θ. But if the angles are so small that they may be replaced by their tangents, we should not be concerned about neglecting their products as compared to unity. If these approximations are accepted then, from Fig. 3.9 the distance Δ_{nm}, which is measured perpendicular to the axis of the unstrained member mn, is

$$\Delta_{nm} = \Delta_n' + \Delta_n'' + \Delta_n''' = A_1\bar{x}' + A_2\bar{x}'' + A_3\,\Delta\bar{x}_3 \tag{3.7}$$

or Δ_{nm} = the statical moment of the M/EI diagram between points m and n about point n

This relationship between the statical moment of the area of the M/EI diagram and the distance Δ_{nm} of point n from the tangent at m is called the second theorem of area moments. The theorem of area moments was given in this form by Charles E. Greene in 1873 and in practically the same manner by Otto Mohr in 1868.

It should be noted that the absolute displacement y_n must still be obtained from the relationship

$$y_n = y_m + \theta_m(\Delta x_1 + \Delta x_2 + \Delta x_3) - \Delta_{nm} \tag{3.8}$$

Therefore the area moments theorems do not give the actual displacement directly unless y_m and θ_m are both zero.

Example 3.2 The beam in Fig. 3.10*a* is restrained at end a such that y_a and θ_a are both zero. Since EI is a constant, the M/EI diagram has a linear variation as shown in Fig. 3.10*b*. In equation 3.5 if point a is used as the reference point m (Fig. 3.9), the expression for the displacement of point c becomes, since y_a and θ_a are zero,

$$y_{ca} = \textstyle\sum_c^a \bar{x}\,\Delta\phi = \sum_c^a \bar{x}\,\Delta A = \Delta_{ca}$$

This summation is represented graphically by the construction in Fig. 3.10*c* and *e* in which the angle changes $\Delta\phi$ and the distances $\bar{x}$ from point c are recorded. These values are obtained from the areas and centroids

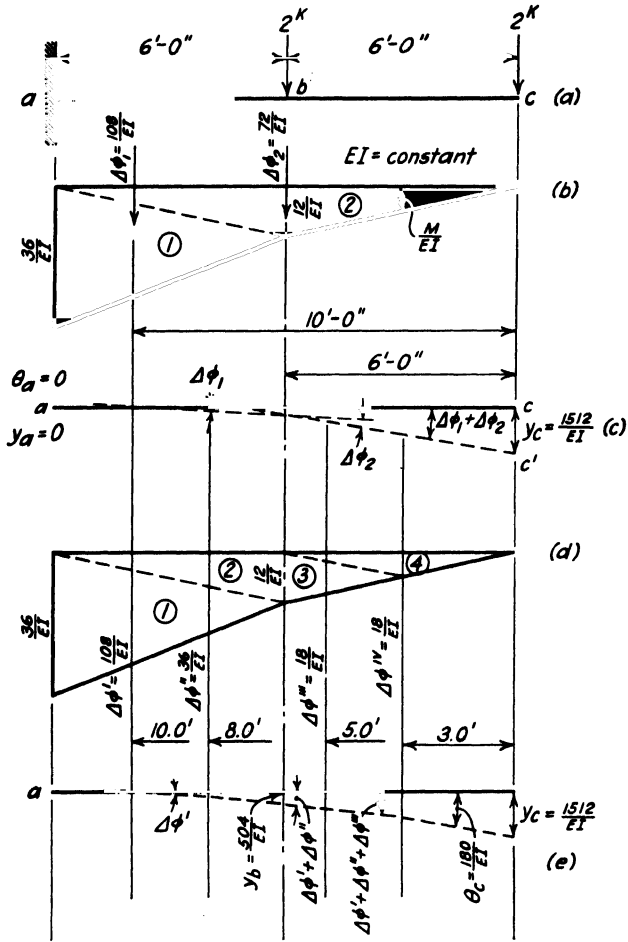



Fig. 3.10

of the M/EI diagram as indicated. It is apparent from Fig. 3.10*c* that the values of y_c and θ_c are

$$y_c = \frac{(108)(10) + (72)(6)}{EI} = \frac{1512}{EI}$$

$$\theta_c = \frac{108 + 72}{EI} = \frac{180}{EI}$$

In this solution only two angle changes, $\Delta\phi_1$ and $\Delta\phi_2$, are used and therefore only y_c and θ_c can be correctly obtained.

To obtain the correct relative displacement and slope with respect to any reference point, the statical moment and area of the M/EI diagram between the two points must be used. Since $\Delta\phi_1$ and $\Delta\phi_2$ in Fig. 3.10*b* are equal to the entire area of the M/EI diagram between *b* and *a*, the displacements and rotations of other points on the elastic curve cannot be obtained from them.

In Fig. 3.10*d*, $\Delta\phi'$ and $\Delta\phi''$ give the correct angle change between *a* and *b* in both magnitude and position and therefore the correct values of y_b and θ_b are

$$y_b = \frac{(108)(4) + (36)(2)}{EI} = \frac{504}{EI}$$

$$\theta_b = \frac{108 + 36}{EI} = \frac{144}{EI}$$

From this example it is apparent that the accuracy with which any particular slope or displacement is obtained from a set of angle changes depends upon the relation of the $\Delta\phi$'s to the area and statical moment of the M/EI diagram between any two points. In general, the true elastic curve will intersect the traverse at one point on each traverse line. Therefore the larger the number of $\Delta\phi$s' used, the closer the traverse approaches the true curve.

3.5 *Conjugate Beam Method (Actual Displacements in Beams)*

The actual displacements in a beam with two nonyielding simple supports are easily calculated by a further development of the slope and moment-area relationship that is commonly designated as the conjugate beam method (Ref. 3.5) or the method of elastic weights (Ref. 4, Chapter 2). This mathematical analogy is readily derived by the geometrical relations illustrated in Fig. 3.11. If we let A equal the area of the M/EI diagram for the entire span *ab*, and if $\bar{x}$ is the distance from *b* to the

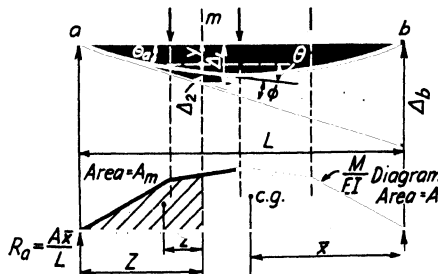



Fig. 3.11

centroid of this area, by equation 3.7, Δ_b is equal to $A\bar{x}$ and the end rotation θ_a is equal to

$$\theta_a = \frac{\Delta_b}{L} = \frac{A\bar{x}}{L} \tag{3.9}$$

The actual rotation θ of the tangent at any section m will be the end rotation θ_a minus ϕ, the relative rotation between sections a and m which is equal to the shaded area A_m between the two sections. Therefore

$$\theta = \theta_a - \phi = \frac{A\bar{x}}{L} - A_m \tag{3.10}$$

If we regard the M/EI diagram as a weight on a beam of span L with supports at a and b, which will be called the conjugate beam, $A\bar{x}/L$, or θ_a, will equal the left reaction of this beam, and the rotation θ is equal to the reaction minus the load A_m, or equals the shear in the conjugate beam at that section.

In addition, the actual displacement y at the section m is equal to

$$y = \Delta_1 - \Delta_2 = (\theta_a)(z) - A_m\bar{z} = \frac{A\bar{x}}{L} z - A_m\bar{z} \tag{3.11}$$

Equation 3.11 states that the ordinate y to the elastic curve of a beam at any section m is numerically equal to the moment of the reaction of the conjugate beam $A\bar{x}/L$ about section m minus the moment of the M/EI diagram between the reaction and the section m. Therefore y equals the bending moment in the conjugate beam.

This analogy can be conveniently summarized as follows. To obtain the rotations and displacements for any beam with two nonyielding supports a and b, construct a conjugate beam by using the M/EI diagram as a load on a similar beam of span ab. Compute the value of the reactions, shear, and bending moment in the conjugate beam for the M/EI loading. From equations 3.10 and 3.11, the shear in the conjugate beam equals the rotation in the actual beam, and the bending moment in the conjugate beam equals the displacement in the actual beam. This analogy holds for any continuous elastic curve between the two nonyielding supports a and b. If the relationship of the M/EI diagram to the geometry of the elastic curve, as previously described, is fully understood, an analogy is not only unnecessary but usually undesirable.

Example 3.3 The end rotations and maximum displacement in beam ab, Fig. 3.12, are computed by the conjugate beam $a'b'$. The area A of the

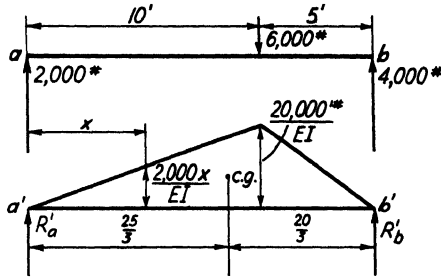



Fig. 3.12

M/EI diagram is

$$A = \frac{1}{2}\left(\frac{20{,}000}{EI}\right)15 = \frac{150{,}000}{EI}$$

$$R_a' = \theta_a = \frac{\left(\dfrac{150{,}000}{EI}\right)\left(\dfrac{20}{3}\right)}{15} = \frac{200{,}000}{3EI} \quad \text{radians}$$

$$R_b' = \theta_b = \frac{\left(\dfrac{150{,}000}{EI}\right)\left(\dfrac{25}{3}\right)}{15} = \frac{250{,}000}{3EI} \quad \text{radians}$$

The value of EI must be in pounds times feet squared.

The maximum displacement in ab will occur where the slope is zero, that is, where the shear in the conjugate beam $a'b'$ is zero. The shear in the conjugate beam at any distance x from a' is

$$V' = R_a' - \frac{1}{2}\,x\left(\frac{2000x}{EI}\right) = \frac{200{,}000}{3EI} - \frac{1000x^2}{EI}$$

and therefore for zero shear

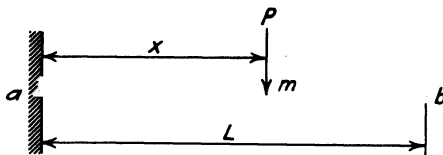
$$x^2 = 66.7 \qquad x = +8.15 \text{ ft}$$

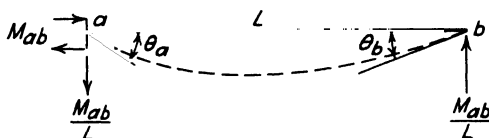
The displacement at this point is equal to the bending moment in the conjugate beam or

$$y_{\max} = \frac{200{,}000}{3EI}\,8.15 - \frac{1}{2}\,8.15\left(\frac{16{,}300}{EI}\right)\left(\frac{8.15}{3}\right) = \frac{362{,}890}{EI} \text{ ft}$$

Problems

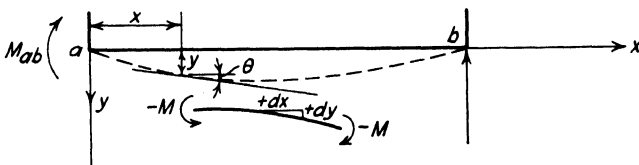
3.6 (*a*) Calculate the rotation and the vertical displacement at points b and m in terms of P, x, and EI by the theorems of area moments.

(*b*) What changes in these values will occur if the tangent at a rotates clockwise through some additional small angle α because of yielding of the support.

Problem 3.6

3.7 Determine by the conjugate beam method the end rotations θ_a and θ_b at points a and b if these points have no vertical displacement. Assume constant EI and show graphically that this problem is similar to the combined effects of Problem 3.6a and b.

Problem 3.7

3.8 If tan θ is equal to dy/dx and $d\theta/dx$ is assumed equal to $d(\tan\theta)/dx$ for small angles, solve Problem 3.7 from the basic relation

$$d\theta/dx = \frac{d(\tan\theta)}{dx} = \frac{d^2y}{dx^2} = -\frac{M}{EI}$$

Note that in the diagram the origin of coordinates is indicated at point a and the positive direction of y is downward. Therefore dy is positive for negative bending moments as shown. These curvature relations should be familiar in both geometric and algebraic form.

Problem 3.8

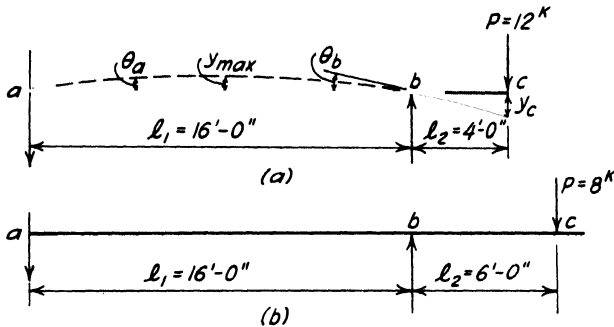
3.9 (a) Express θ_a, θ_b, $y_{\max}$, and y_c in terms of EI which is constant.

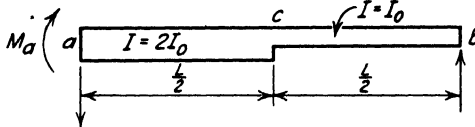
(b) Show that the elastic curve for the portion ab for a load P of 8 kips and l_2 of 6 ft is the same as in part a for 12 kips and 4 ft respectively.

(c) Show by the principle of superposition that

$$\theta_b = \frac{M_b l_1}{3EI} = \frac{Pl_2 l_1}{3EI}$$

and

$$y_c = \theta_b l_2 + \frac{Pl_2^3}{3EI} = \frac{Pl_2^2}{3EI}(l_1 + l_2)$$

Problem 3.9

3.10 Determine the rotations of the tangents at a and b in terms of M_a, L, and EI_0.

Problem 3.10

3.11 Calculate the rotations θ_a and θ_b and the vertical displacement Δ_c for a vertical load P at the center of the beam in Problem 3.10 if M_a is removed.

3.6 *Calculation of Redundant Forces from Specified Beam Displacements*

The numerical calculations of displacements in the preceding examples have been made for statically determinate beams. The elastic curve of a beam can be completely determined by these methods for any known force system. However, there is no reason why the displacements cannot be expressed in terms of any number of forces, known or unknown, by means of equations 3.7, 3.8, 3.10, 3.11. The development and use of such equations are treated in more detail in succeeding chapters, but a few applications can be discussed conveniently now. If certain strain conditions, as discussed in Chapter 1, are known, one redundant force can be determined for each specified strain condition. These strain conditions can be expressed in terms of either rotations or translations.

When drawing the M/EI diagram for general force systems, a convenient arrangement is to separate the known loads and their reactions from the redundant forces and their reactions. This resolution of forces is permissible if the principle of superposition can be applied. Any displacement is then expressed in terms of both the given loading and the unknown forces, from which the unknown forces can be calculated so as

to satisfy any known strain conditions. *The numerical procedure is best explained by the following examples.*

Example 3.4 The fixed-end moment M_{Fab} for the beam *abc*, Fig. 3.13*a*, will be determined for a uniform load of 2 kips per foot (EI = constant). As previously defined, a fixed-end moment is an end couple that will prevent rotation of the cross section at *a*.

For convenience, the M/EI diagram for the span *ab* is drawn in three parts: (1) for the uniform load on the span *ab*, Fig. 3.13*b*, (2) for the uniform load on the cantilever portion *bc*, Fig. 3.13*c*, (3) for the fixed-end moment M_{Fab} which is assumed positive, Fig. 3.13*d*. These three diagrams constitute the load on the conjugate beam, and, if the tangent at *a* is to remain horizontal ($\theta_a = 0$), the shear R_a' in the conjugate beam from this loading must be zero or, taking the statical moments of the M/EI diagrams about point *b*,

$$16R_a' = \frac{2}{3}\frac{64}{EI}(16)(8) - \frac{1}{2}\frac{16}{EI}(16)\left(\frac{16}{3}\right) + \frac{1}{2}\frac{M_{Fab}}{EI}(16)\left(\frac{32}{3}\right) = 0$$

from which

$$M_{Fab} = -56 \text{ ft-kips}$$

The negative sign shows that the assumed direction is wrong. The reaction R_b is now known as

$$R_b = 16 + 9 - \frac{56}{16} = 21.5 \text{ kips}$$

Example 3.5 The fixed-end moments M_{Fab} and M_{Fba} (Fig. 3.14*a*) will be calculated for the concentrated load of 10 kips at the center of the span and for the variation in I as shown. Again the M/EI diagram is drawn for three different force systems: (1) for the concentrated load of

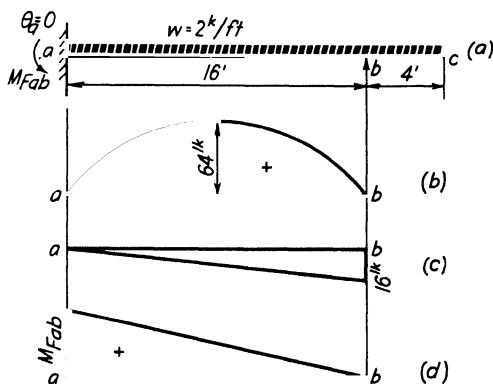



Fig. 3.13

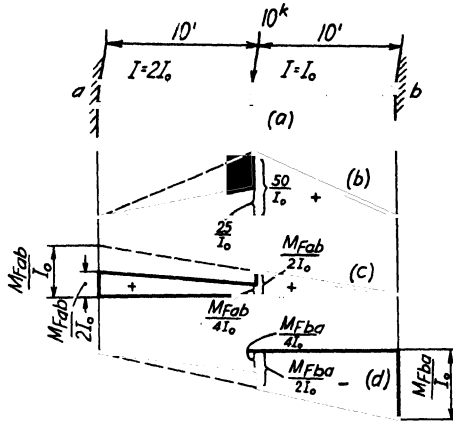



Fig. 3.14

10 kips (Fig. 3.14*b*), (2) for the fixed-end moment M_{Fab}, assumed clockwise (Fig. 3.14*c*), (3) for the fixed-end moment M_{Fba}, assumed clockwise (Fig. 3.14*d*). In this problem the shear at both ends of the conjugate beam must be zero ($\theta_a = \theta_b = 0$). Consequently, the statical moment of the combined M/EI diagram about both a and b must be zero,

$$\sum M_b = 0 \qquad \left(\frac{1}{EI_0} \text{ is omitted from all terms}\right)$$

$$125\left(\frac{40}{3}\right) + 250\left(\frac{20}{3}\right) + M_{Fab}\left[1.25\left(\frac{50}{3}\right) + (2.5)(15) + \frac{(2.5)(20)}{3}\right]$$

$$- M_{Fba}\left[1.25\left(\frac{40}{3}\right) + (5)(5) + 2.5\left(\frac{10}{3}\right)\right] = 0$$

$$\sum M_a = 0$$

$$125\left(\frac{20}{3}\right) + 250\left(\frac{40}{3}\right) + M_{Fab}\left[1.25\left(\frac{10}{3}\right) + (2.5)(5) + 2.5\left(\frac{40}{3}\right)\right]$$

$$- M_{Fba}\left[1.25\left(\frac{20}{3}\right) + (5)(15) + 2.5\left(\frac{50}{3}\right)\right] = 0$$

or

$$22.5M_{Fab} - 15.0M_{Fba} = -1000$$

$$15.0M_{Fab} - 37.5M_{Fba} = -1250$$

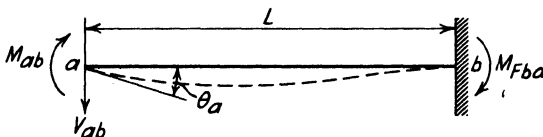
from which

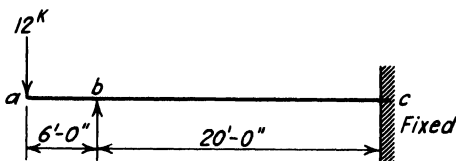
$$M_{Fab} = -30.3 \text{ ft-kips} \qquad M_{Fba} = 21.2 \text{ ft-kips}$$

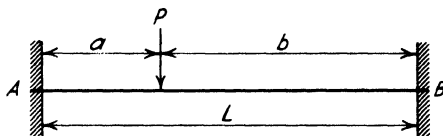
The negative sign for M_{Fab} shows that it is counterclockwise instead of clockwise as assumed.

Problems

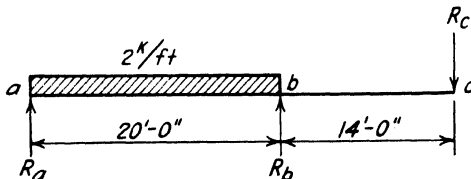
3.12 (*a*) If Δ_a, Δ_b, and θ_b are zero and EI is constant, determine the values of V_{ab}, M_{Fba}, and θ_a in terms of M_{ab}, L, and EI.

(*b*) Determine the maximum vertical displacement in the span.

Problem 3.12

3.13 Determine the value of the reaction at *b* and the moment at *c* by the solution of Problem 3.12.

Problem 3.13

3.14 Solve for the end couples M_{Fab} and M_{Fba} by the theorem of area moments.

Problem 3.14

3.15 (*a*) Determine the reactions R_a, R_b, and R_c. Draw the shear and bending-moment diagrams. EI is constant.

(*b*) Calculate the vertical displacement at the center of span *ab* and of span *bc* if the value of EI is 9×10^6 in^2-kips.

Problem 3.15

3.16 Solve Problem 3.15 if end *a* is fixed (no rotation or translation).

3.17 Calculate the fixed-end moment M_{Fab} and the rotation of the tangent at *b* for a moment M_{ba} applied at end *b* and variation in I as shown.

Problem 3.17

3.18 Determine the reactions for the beam in Problem 3.17 for a vertical load of 16 kips at the center of the span and for a span L of 16 ft. Remove the end couple M_{ba}, but keep end a fixed.

3.19 Plot the change in the shear and bending-moment diagrams for the beam in Problem 3.16 if support b moves downward 0.3 in. Use EI equal to 9×10^6 in²-kips.

3.7 *Further Applications of Angle Changes*

In Articles 3.4 and 3.5 the displacements in straight beams because of angle changes $\Delta\phi$ are obtained from the products of $\Delta\phi$ and the distance from the section where the resultant angle change occurs to the section where the displacement is measured. All distances are measured along the original axis of the beam and all displacements are perpendicular to it. When the structure is composed of several members whose axes intersect at various angles, as in Fig. 3.15, a more general conception of

Fig. 3.15

the geometric relations between displacements and angle changes is necessary. These relationships will be illustrated by considering the horizontal and vertical displacements of point a in Fig. 3.15 because of any angle change $d\phi$ at some section n.

For a small rotation $d\phi$ at section n, the movement $d\,\Delta_a$ at point a is perpendicular to the line an and the magnitude is

$$d\,\Delta_a = r\,d\phi \tag{3.12a}$$

The absolute values of the vertical and horizontal components of this displacement are,

$$d\,\Delta_{av} = d\,\Delta_a \sin\beta = r\,d\phi\frac{x}{r} = x\,d\phi \tag{3.12b}$$

and

$$d\,\Delta_{ax} = d\,\Delta_a \cos\beta = r\,d\phi\frac{y}{r} = y\,d\phi \tag{3.12c}$$

From equations 3.12*b*, *c* it can be seen that the vertical displacement of any point because of an angle change at another point on a continuous elastic curve is equal to the angle change times the horizontal distance between the two points; also the horizontal displacement is equal to the angle change times the vertical distance between the two points.

As in previous problems the rotations and displacements at the boundaries of any structure must be considered. Thus in Fig. 3.15 the resultant horizontal displacement Δ_{ax} of point a with respect to d is

$$\Delta_{ax} = \Delta_{dx} - c\alpha_d - \int_d^a y\,d\phi \tag{3.13a}$$

and the resultant vertical displacement is

$$\Delta_{av} = \Delta_{dv} + L\alpha_d + \int_d^a x\,d\phi \tag{3.13b}$$

In the preceding equations x, y, Δ_{ax}, and Δ_{av} have been assumed positive when acting to the right or upward, whereas α_d and $d\phi$'s are assumed positive if clockwise. A definite sign convention must be established before any numerical operations are started. The integrals involving $x\,d\phi$ and $y\,d\phi$ can be evaluated from the statical moment of the M/EI diagram. This procedure will be explained by a numerical example.

Example 3.6 The horizontal displacement of point a in Fig. 3.16 will be determined from equation 3.13*a* for the condition that Δ_{av}, Δ_{dx}, and Δ_{dv} are all zero. The following substitutions will be made for the integrals

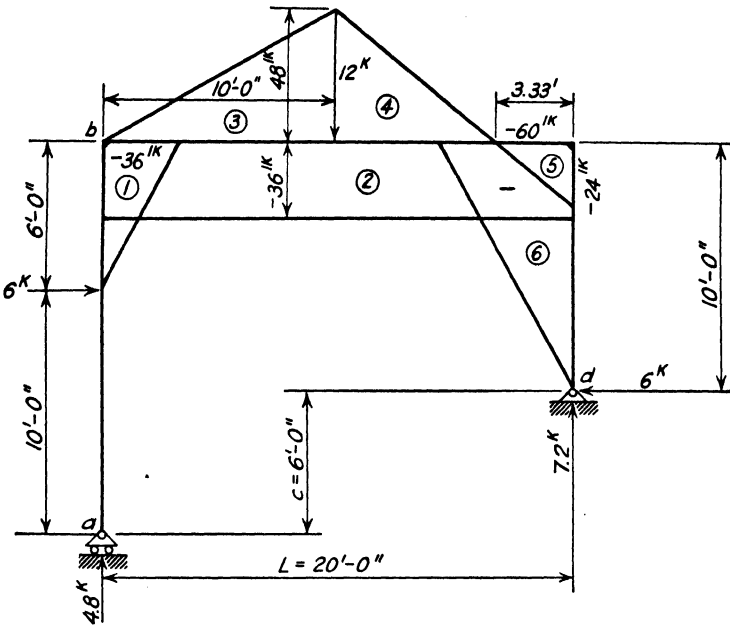



Fig. 3.16

in equations 3.13*a* and 3.13*b*.

$$\int_d^a y\, d\phi = \sum_d^a A\bar{y} \tag{3.14a}$$

and

$$\int_d^a x\, d\phi = \sum_d^a A\bar{x} \tag{3.14b}$$

where $\Sigma\, A\bar{y}$ and $\Sigma\, A\bar{x}$ are the statical moments of the M/EI diagram about x and y axes through point a. The numerical calculations for these values are summarized in Table 3.1.

From equation 3.13*b*, if Δ_{ay} and Δ_{dy} are zero, we obtain

$$\Delta_{ay} = 0 = 0 + 20\alpha_d - 10{,}400$$

or, assuming EI equal to unity,

$$\alpha_d = \frac{10{,}400}{20} = 520$$

From equation 3.13*a*, if Δ_{dx} is zero, the value of Δ_{ax} is

$$\Delta_{ax} = -(6)(520) - (-11{,}050)$$

or

$$\Delta_{ax} = 7930$$

Table 3.1 (*EI is omitted*)

Bending-Moment Area No.	A (+↻)	$\bar{x}$ (+→)	$\bar{y}$ (+↑)	$A\bar{x}$	$A\bar{y}$
1	−108	0	14.00	0	−1512
2	−720	10.00	16.00	−7200	−11500
3	+240	6.67	16.00	+1600	+3840
4	+160	12.22	16.00	+1955	+2560
5	−40	18.89	16.00	−755	−640
6	−300	20.00	12.67	−6000	−3800
				−10,400	−11,050

The actual values are

$$\alpha_d = \frac{520}{EI} \qquad \text{and} \qquad \Delta_{ax} = \frac{7930}{EI}$$

Example 3.7 The displacement Δ_{ax} for a unit horizontal load applied at point a in the frame used in Example 3.6 is determined for the same boundary conditions, that is, Δ_{av}, Δ_{dx}, and Δ_{dv} are zero. The bending-moment diagram is shown in Fig. 3.17 and the statical moments of the M/EI diagram (assuming EI is constant) are given in Table 3.2.
From equation 3.13*b*,

$$\Delta_{av} = 0 + 20\alpha_d + 3400 = 0$$

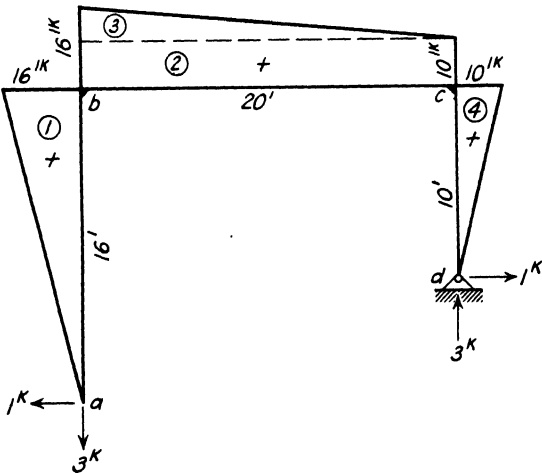



Fig. 3.17

Table 3.2

Bending-Moment Area No.	A	$\bar{x}$	$\bar{y}$	$A\bar{x}$	$A\bar{y}$
1	128	0	10.67	0	1367
2	200	10.00	16.00	2000	3200
3	60	6.67	16.00	400	960
4	50	20.00	12.67	1000	633
				3400	6060

or

$$\alpha_d = -170 \qquad \left(\text{actually } -\frac{170}{EI}\right)$$

and

$$\Delta_{ax} = 0 - (6)(-170) - 6060 = -5040$$

or

$$\Delta_{ax} = -\frac{5040}{EI}$$

Example 3.8 From the results of Examples 3.6 and 3.7 the horizontal reaction H_a in Fig. 3.18 will be determined for the strain condition that Δ_{ax} is zero. The total displacement Δ_{ax} for all forces can be written in

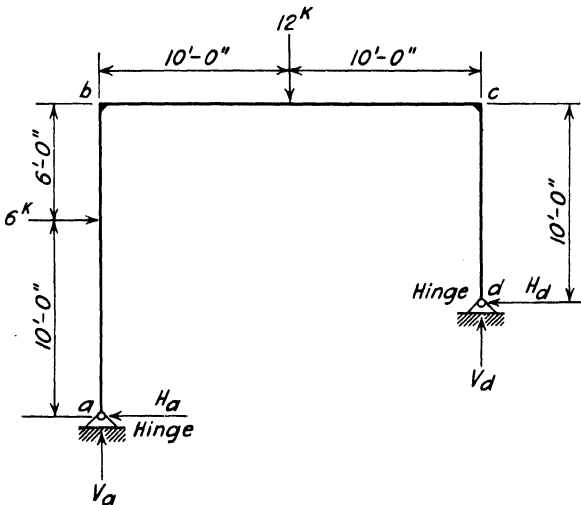



Fig. 3.18

the form

$$\Delta_{ax} = \frac{7930}{EI} + H_a\left(-\frac{5040}{EI}\right) = 0$$

or

$$H_a = \frac{7930}{5040} = 1.57 \text{ kips}$$

3.8 *Relative Displacements in Frames*

A Williot diagram can also be used to obtain the relative displacements of the ends of members that undergo considerable flexure but have small axial stresses. This condition usually exists in rigid-frame structures, and a study of such displacements is an important factor in their analysis. For example, if the frame *abcd* (Fig. 3.19*a*) is moved to the right by the load P, the displacements of the joints are largely because of the internal strains that are caused by bending moments, instead of axial stress as in a truss. If the change in length of the members is neglected, point b must be somewhere on arc 1-1, which has the fixed point a for a center and ab as radius. Also point c must be somewhere on arc 2 whose radius is cd and arc 3 whose radius is bc and center b'. As the rotations of the various members are small, the arcs can be replaced by perpendiculars as for trusses. From the construction it is apparent that the triangle $cc'b''$ gives the movement of b with respect to a, and c with respect to b and d. If this diagram is drawn out separately, as in Fig. 3.19*b*, it is identical in construction with a Williot diagram in which the elongations are all zero. Thus $a'b'$ is perpendicular to ab, $c'd'$ to cd, $b'c'$ to bc. All displacements can be expressed in terms of Δ, the horizontal movement, as follows.

$$\begin{aligned} a'b' &= bb' = \Delta \sec \phi_1 \\ c'd' &= cc' = \Delta \sec \phi_2 \\ b'c' &= \Delta(\tan \phi_1 + \tan \phi_2) \end{aligned} \tag{3.15}$$

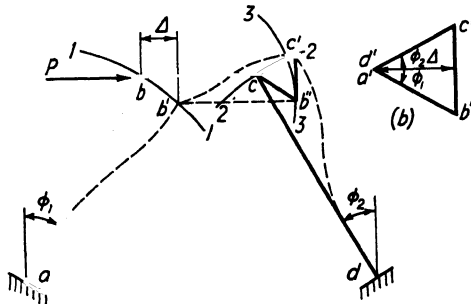



Fig. 3.19 **Relative displacements in a quadrangular frame.**

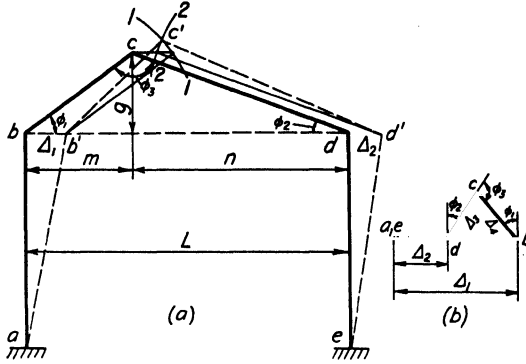



Fig. 3.20 Relative displacement in a gable frame.

The value of Δ can be ascertained by methods that will be explained later.

The same procedure can be followed for the gable frame in Fig. 3.20*a* in which b and d must move horizontally to b' and d' respectively. Point c then moves to c', which is on the intersection of arcs 1-1 and 2-2. If the arcs are replaced by perpendiculars, the movement of all joints can be shown by the displacement diagram of Fig. 3.20*b*, which is simply the construction at c in Fig. 3.20*a* to a larger scale. From the geometrical relations of the displacement diagram,

$$\frac{\Delta_4}{\sin(90° - \phi_2)} = \frac{\Delta_1 - \Delta_2}{\sin(180° - \phi_3)}$$

or

$$\Delta_4 = (\Delta_1 - \Delta_2)\frac{\cos\phi_2}{\sin(180° - \phi_3)}$$

But

$$\cos\phi_2 = \frac{n}{cd}$$

and

$$\sin(180° - \phi_3) = \frac{L\sin\phi_1}{cd} = \frac{Lg}{(cd)(bc)}$$

from which the rotation of member bc is equal to

$$\frac{\Delta_4}{bc} = (\Delta_1 - \Delta_2)\frac{n}{gL} \tag{3.16}$$

Similarly, the rotation of cd which is equal to Δ_3/cd is obtained from the relation

$$\frac{\Delta_3}{\sin(90° - \phi_1)} = \frac{\Delta_1 - \Delta_2}{\sin(180° - \phi_3)}$$

which, by the relations already established, is equal to

$$\frac{\Delta_3}{cd} = (\Delta_1 - \Delta_2)\frac{m}{gL} \tag{3.17}$$

In the analysis of such frames, it is therefore necessary to consider only the two independent displacements Δ_1 and Δ_2.

3.9 Newmark's Method for Calculating Angle Changes

N. M. Newmark in 1943 (Ref. 8) presented a convenient and accurate method for determining the area of most M/EI diagrams in terms of concentrated values ϕ, so arranged that they will also provide an accurate determination of the statical moment of the area. If any curve can be

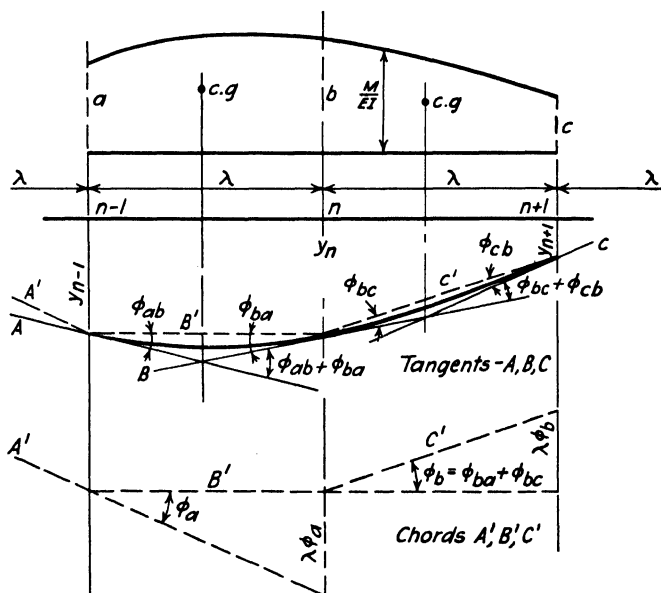



Fig. 3.21

defined by a number of ordinates, the area under the curve can usually be closely approximated by assuming that any portion of the curve connecting three consecutive points is a parabola. This area can, of course, be obtained by Simpson's rule which is based on the same assumption. However, in Newmark's method both the magnitude and distribution of the concentrations are made statically equivalent to the assumed area. Therefore, as shown in Fig. 3.21, the angle changes between the chords, which are now designated by ϕ, can be obtained from the formulas given in the diagram. The relationship between these values of ϕ and the corresponding angle changes of the tangents is illustrated in Fig. 3.21. The principles involved here are similar to the geometrical relations discussed in Article 3.4 except that the angle changes of the chords are used instead of the tangents. The advantages of this procedure can be best illustrated by a numerical example.

Example 3.8 The end rotations θ_0 and θ_6 and the deflections at points $L/6$ apart will be determined for a uniform load of w lb per unit length acting on the beam shown in Fig. 3.22. The values of the bending moments M in terms of wL^2 and of I in terms of I_0, the moment of inertia at point 0, are recorded on the diagram for each point from zero to six. From these values the magnitudes of M/EI in terms of wL^2/EI_0 were calculated and recorded. From the expressions for the area concentrations of the M/EI diagram given in Fig. 3.21, the following chord angle changes ϕ are obtained.

$$\phi_0 = \frac{wL^3}{72EI_0}\left[(3.5)(0) + (3)(0.0522) - \left(\frac{1}{2}\right)(0.0643)\right] = 0.1245C$$

$$\text{where } C = \frac{wL^3}{72EI_0} = \text{the multiplier}$$

$$\phi_1 = C[0 + 0.5220 + 0.0643] = 0.5863C$$

$$\phi_2 = C[0.0522 + 0.643 + 0.0568] = 0.7520C$$

$$\phi_3 = C[0.0643 + 0.568 + 0.0406] = 0.6729C$$

$$\phi_4 = C[0.0568 + 0.406 + 0.0206] = 0.4834C$$

$$\phi_5 = C[0.0406 + 0.206 + 0] = 0.2466C$$

$$\phi_6 = C[-(\tfrac{1}{2})(0.0406) + (3)(0.0206) + 0] = 0.0415C$$

If the tangent θ_6 at point 6 is assumed to be zero, the position of the elastic curve is defined by the position of the chords as shown by the

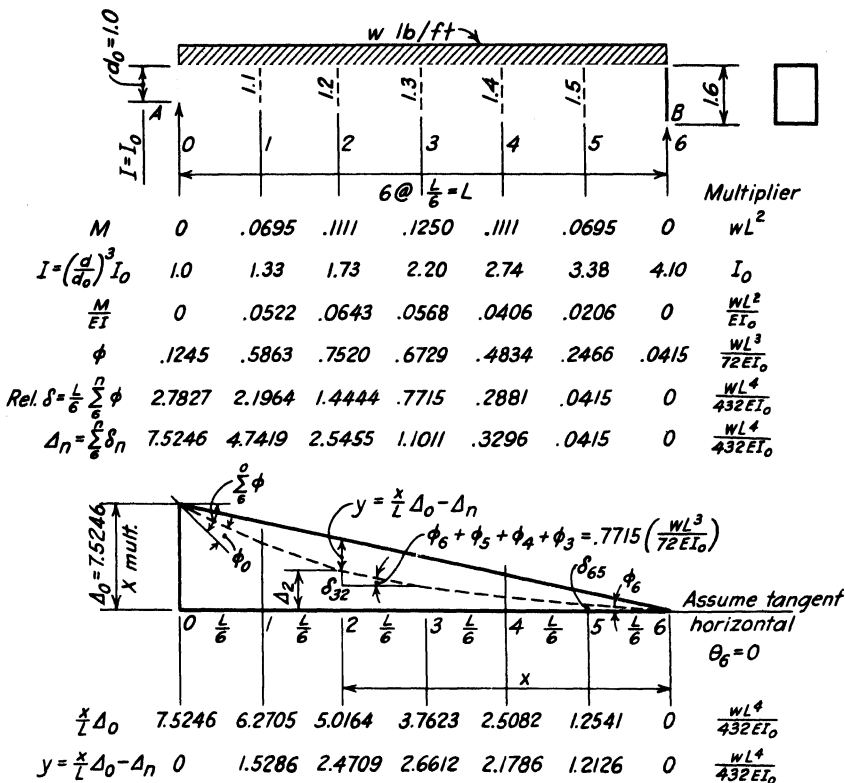



Fig. 3.22

broken line in Fig. 3.22. The relative deflection δ between any two consecutive points is equal to the rotation of the chord connecting the two points from the reference tangent θ_6 times the distance $L/6$. However, the rotation of the chord is equal to the summation of the ϕ values from the reference end, that is, point six. The reader should calculate all ϕ and δ values and understand their geometrical meaning. The total displacement Δ from the reference tangent θ_6 is obviously equal to the summation of all δ values from the reference point. The values of Δ with respect to the tangent at point six are recorded on the diagram in Fig. 3.22. Since the value of Δ_0 is 7.5246 $(wL^4/432EI_0)$, then to give no relative displacement between points six and zero, the beam must be rotated through a counterclockwise angle θ_6 at point six such that

$$\theta_6 = \tan\theta_6 = -\frac{(7.5246)(wL^4/432EI_0)}{L}$$

The actual deflection y of any point a distance x from point six as shown in Fig. 3.22 is

$$y = \frac{x}{L}\Delta_0 - \Delta_n$$

The values of y have been tabulated on the diagram.

The rotation θ_0 at point zero is

$$\theta_0 = \theta_6 + \textstyle\sum_6^0 \phi = -7.5246\frac{wL^3}{432EI_0} + 2.9072\frac{wL^3}{72EI_0}$$

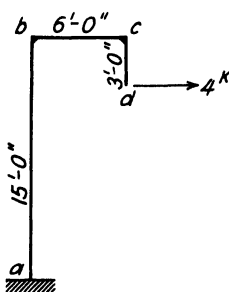
or

$$\theta_0 = 0.0229\frac{wL^3}{EI_0}$$

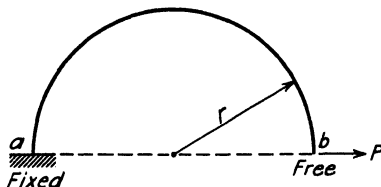
Problems

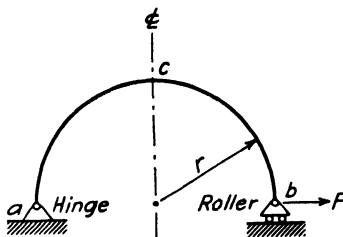
3.20 (*a*) Determine the horizontal displacement of point d in terms of EI.

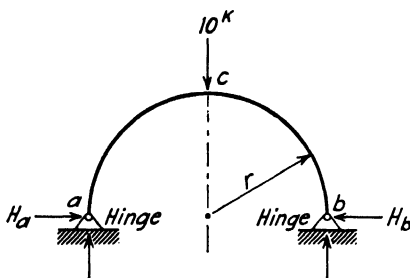
(*b*) What is the value of the vertical displacement of d?

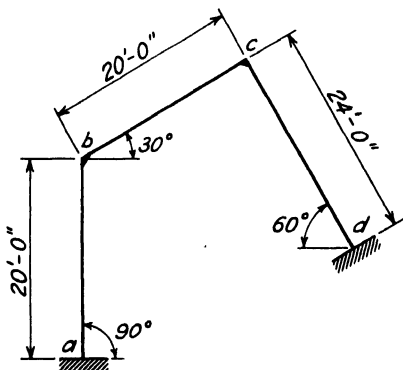
(*c*) Compare the equations that you used for the solution of parts (*a*) and (*b*) with the corresponding equations obtained by the method of virtual work and by Castigliano's theorem.

Problem 3.20

3.21 (*a*) What is the amount of the horizontal displacement of point b if EI is a constant? Use the principle that horizontal displacement due to flexure is equal to angle changes times vertical distance.

(*b*) Calculate the vertical displacement of point b.

Problem 3.21

3.22 Determine the horizontal displacement of b and the vertical displacement of c in terms of F, r, and EI.

Problem 3.22

3.23 From the results of Problem 3.22 calculate the value of H_b if the two arches are identical except for the support at b.

Problem 3.23

3.24 If joint b moves 10 units horizontally to the right, calculate the rotations of members bc and cd due to flexure only.

Problem 3.24

3.25 Solve Example 3.8 if the uniform load is replaced by a concentrated load P at the center of the span.

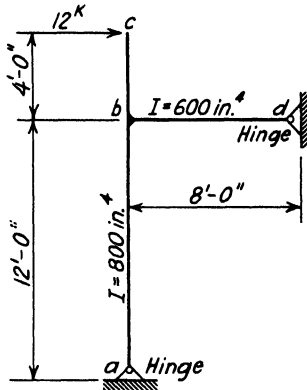
3.26 Calculate the vertical deflections and end rotations for the simply supported beam in Example 3.8 due to a clockwise couple of 1000 ft-lb applied at the center of the span. Express all values in terms of EI_0.

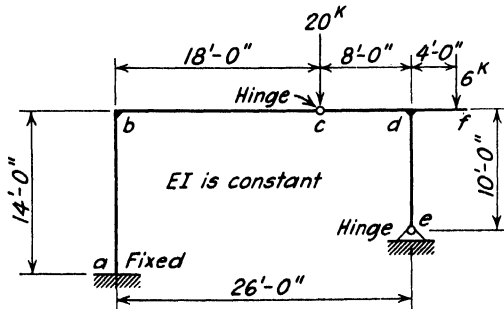
3.27 Prepare a computer program that will give the numerical values of the end rotations and the vertical displacements at n points in a simply supported beam. Assume that the numerical values of n, L, M, and I will be provided as input data. Use the Newmark procedure for determining angle changes as explained in Article 3.9. Check your program from the numerical data and results given in Example 3.8.

3.28 (*a*) Solve for the reactions of the frame in Problem 2.25 by the area-moment method.

(*b*) Calculate the horizontal displacement of point c by use of angle changes.

3.29 (*a*) Determine the reactions at a and d for the frame shown by the area-moment method.

(*b*) Express the horizontal displacement at c in terms of E.

Problem 3.29

3.30 Calculate the reactions at hinge e and the vertical displacement of hinge c for the frame shown. Express in terms of EI which is constant.

Problem 3.30

References

1 C. E. Greene, *Trusses and Arches*, John Wiley and Sons, 1891.
2 Williot, "Notations pratiques sur la statique graphique," *Publ. Scientifiques Industrielles*, 1877.
3 G. E. Beggs, "The Use of Models in the Solution of Indeterminate Structures," *J. Franklin Institute*, p. 203, 1927.
4 Otto Gottschalk, "Structural Analysis Based upon Principles Pertaining to Unloaded Models," *Trans. Am. Soc. C. E.*, Vol. 103 (1938).
5 H. M. Westerguard, "Deflection of Beams by the Conjugate Beam Method," *J. Western Soc. Engrs.*, November 1921.
6 J. S. Kinney, *Indeterminate Structural Analysis*, Addison-Wesley Publishing Company, 1957.
7 J. Michalos, *Theory of Structural Analysis and Design*, Ronald Press, 1958.
8 N. M. Newmark, "Numerical Procedure for Computing Deflections, Moments, and Buckling Loads," *Trans. Am. Soc. C. E.*, Vol. 108 (1943).

4

Continuous Beams and Frames with Straight Prismatic Members

DERIVATION AND APPLICATION OF SLOPE-DEFLECTION EQUATIONS

4.1 Definitions and Assumptions

The term continuity is applied to structures when the members are so connected that moments, torques, shears, and thrusts can be transmitted from one member or unit to another. The building frame in Fig. 4.1 illustrates this type of framing, which has been used in reinforced-concrete construction for many years and is now frequently employed in steel structures. When there is practically no deformation in the connection itself, the continuous frame is often called a rigid frame. The characteristic action of such a structure is the restraint or assistance that the columns give to the beams when the beams are subjected to vertical loads and the restraint that the beams give to the columns when the frame is subjected to horizontal loads, such as those caused by wind or earthquake action.

The analysis of continuous frames will begin by studying certain mathematical relations between forces and displacements that are essential for a satisfactory understanding of recent analytical methods. The most convenient displacements to use in the solution of many continuous frame structures are the rotations θ and the translations Δ (Fig. 4.2) of the various joints. The term joint denotes the material at the intersection of several members that is common to all. It cannot be

Fig. 4.1 Rigid-frame construction.

described exactly. A common example is given in Fig. 4.1, where each member of the frame is considered to be of constant cross section up to the intersection of the dotted lines that represent the axes of the members. Such an assumption will reduce the joint to a point, a conception that is useful for analytical purposes but which must be modified in the final design. If the elastic curve of each member is assumed to be continuous to the intersection of the axes and no deformation takes place within the joint, the rotation of the end tangent must be the same for each member, as, for example, θ_a in Fig. 4.1. This rotation can also be described as the joint rotation. Any position of the frame can be defined in terms of the displacements of the joints and the elastic curves of the members.

4.2 End Forces and Sign Conventions

The coplanar end forces acting on any member of a continuous frame such as *ab* in Fig. 4.1 are illustrated in Fig. 4.3. Thus at end *a*, the forces that are developed from the action of any applied coplanar force system are: an end couple M_{ab}, a normal force N_{ab}, and a shear force V_{ab}. Similar forces are acting at section *b* or on any transverse section through the member. To describe adequately any force in algebraic computations, it is necessary to assign a definite direction, magnitude, and sense, and therefore any sign convention must provide this information.

As an example of sign conventions, the positive direction for N_{ab} in Fig. 4.3 can be taken to the right; positive V values are assumed upward and positive couples are assumed clockwise. The sign convention should be established for the forces acting on the members and not on the joints.

Fig. 4.2

In general, the sign conventions for linear displacements should be consistent with those for forces, and the positive direction for rotations should be clockwise as it is for the couples. Generally, all unknowns should be assumed in a positive direction. The particular sign convention that is used is not so important

Fig. 4.3

as the consistent use of it. Unless stated otherwise, the positive direction for end rotations and end couples will be clockwise regardless of which end is considered. The positive direction for end forces and displacements will be designated when used.

4.3 End Couples Expressed in Terms of End Rotations

If the deformations due to axial and shearing forces are neglected, the relationship between forces and displacements because of flexural action can be evaluated from the angle changes as described in Articles 3.4 and 3.5 or from the elastic energy methods as explained in Articles 2.9 and 2.10. Regardless of the algebraic solution used, the relation of forces to the corresponding elastic curve depends on the characteristics of the M/EI diagram for those forces.

In the following derivations the value of EI is considered constant and the variation of any displacement with respect to the forces is linear. Therefore the principle of superposition can be applied and the effects of any number of force systems can be treated separately and then combined algebraically. A force system includes both the applied force and the reactions. A redundant force is treated as an applied force of unknown magnitude.

The importance of the principle of superposition is illustrated in the following examples. The end rotations θ_a' and θ_b' due to the end couple M_{ab} (assumed positive) in Fig. 4.4*a* can be readily obtained from equations 2.33 and 2.34. These values are

$$\theta_a' = \frac{\frac{1}{2}(M_{ab})(L)\frac{2}{3}L}{LEI} = \frac{M_{ab}L}{3EI}$$

$$\theta_b' = -\frac{M_{ab}L}{6EI}$$

For the force system in Fig. 4.4*b* the end rotations due to M_{ba} (assumed positive) are

$$\theta_a'' = -\frac{M_{ba}L}{6EI}, \qquad \theta_b'' = \frac{M_{ba}L}{3EI}$$

(a)

(b)

(c)

(d)

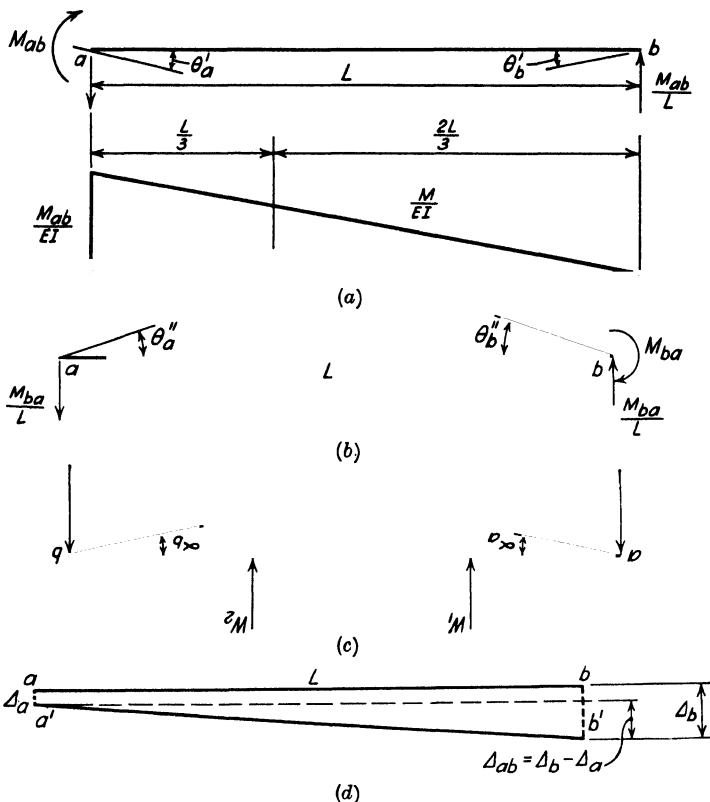



Fig. 4.4

The rotations θ_a and θ_b when both M_{ab} and M_{ba} are applied are therefore

$$\theta_a = \frac{M_{ab}L}{3EI} - \frac{M_{ba}L}{6EI} \tag{4.1a}$$

$$\theta_b = -\frac{M_{ab}L}{6EI} + \frac{M_{ba}L}{3EI} \tag{4.1b}$$

If the equations 4.1*a*, *b* are solved for the end couples in terms of the end rotations, we obtain

$$M_{ab} = 4\frac{EI}{L}\theta_a + 2\frac{EI}{L}\theta_b \tag{4.2a}$$

$$M_{ba} = 2\frac{EI}{L}\theta_a + 4\frac{EI}{L}\theta_b \tag{4.2b}$$

In deriving equations 4.1*a*, *b*, no transverse forces were applied to the beam. However, to take such forces into consideration, as in Fig. 4.4*c*, it

is necessary only to add the angles α_a and α_b with their proper signs. Assuming both angles positive (α_b will usually be negative), the resultant angles at a and b for all three force systems Figs. 4.4a, b, c are

$$\theta_a = \theta_a' - \theta_a'' + \alpha_a \tag{4.3a}$$

$$\theta_b = -\theta_b' + \theta_b'' + \alpha_b \tag{4.3b}$$

If equations 4.3a, b are again solved for M_{ab} and M_{ba}, we obtain

$$M_{ab} = \frac{EI}{L}(4\theta_a + 2\theta_b) - \frac{EI}{L}(4\alpha_a + 2\alpha_b) \tag{4.4a}$$

$$M_{ba} = \frac{EI}{L}(2\theta_a + 4\theta_b) - \frac{EI}{L}(2\alpha_a + 4\alpha_b) \tag{4.4b}$$

If equations 4.4a, b are studied carefully, it is apparent that they will provide the solutions to the force systems shown in Fig. 4.5a, b, c. Thus in Fig. 4.5a in which θ_b, Δ_a, Δ_b, α_a, and α_b are all zero,

$$M_{ab}' = 4\frac{EI}{L}\theta_a \tag{4.5a}$$

and

$$M_{ba}' = 2\frac{EI}{L}\theta_a = \frac{M_{ab}'}{2} \tag{4.5b}$$

(a)

(b)

(c)

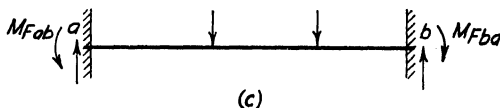
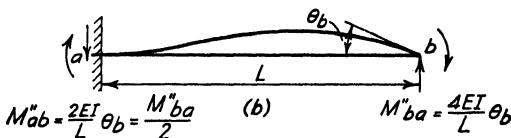
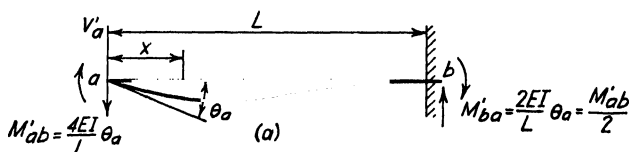



Fig. 4.5

In Fig. 4.5*b* the end couples are

$$M_{ab}'' = 2\frac{EI}{L}\theta_b = \frac{M_{ba}''}{2} \tag{4.6a}$$

$$M_{ba}'' = 4\frac{EI}{L}\theta_b \tag{4.6b}$$

In Fig. 4.5*c* for which θ_a and θ_b are zero the end couples (usually called fixed-end couples) are

$$M_{Fab} = -\frac{EI}{L}(4\alpha_a + 2\alpha_b) \tag{4.7a}$$

$$M_{Fba} = -\frac{EI}{L}(2\alpha_a + 4\alpha_b) \tag{4.7b}$$

Equations 4.7*a*, *b* indicate that the fixed-end couples can be considered as end couples that will rotate the end tangents through angles of minus α_a and minus α_b, that is, will rotate the tangents in Fig. 4.4*c* back to their original position. Probably the most important feature of this discussion is the concept of varying boundary forces to control boundary rotations and translations.

4.4 *Effect of End Translation*

When the ends of any member *ab* are also displaced transversely by a small relative movement Δ_{ab} (Fig. 4.4*d*), both end sections are rotated through an angle that is practically equal to Δ_{ab}/L. Since this condition can be superimposed upon the rotations already considered, it is necessary to rewrite equations 4.3*a*, *b* in the more general form

$$\theta_a = \theta_a' - \theta_a'' + \frac{\Delta_{ab}}{L} + \alpha_a \tag{4.3c}$$

$$\theta_b = -\theta_b' + \theta_b'' + \frac{\Delta_{ab}}{L} + \alpha_b \tag{4.3d}$$

If θ_a', θ_a'', θ_b' and θ_b'' are again expressed in terms of M_{ab} and M_{ba}, equations 4.3*c*, *d* will provide the following solution for M_{ab} and M_{ba} in a more general form.

$$M_{ab} = M_{Fab} + \frac{EI}{L}\left(4\theta_a + 2\theta_b - \frac{6\Delta_{ab}}{L}\right) \tag{4.4c}$$

$$M_{ba} = M_{Fba} + \frac{EI}{L}\left(2\theta_a + 4\theta_b - \frac{6\Delta_{ab}}{L}\right) \tag{4.4d}$$

It should be noted that the rotation Δ_{ab}/L of the axis of the member has been assumed to be clockwise or positive. Sometimes equations

4.4*c*, *d* are written in another form by simply combining the respective rotations at each end, that is,

$$M_{ab} = M_{Fab} + \frac{4EI}{L}\left(\theta_a - \frac{\Delta_{ab}}{L}\right) + \frac{2EI}{L}\left(\theta_b - \frac{\Delta_{ab}}{L}\right) \qquad (4.4e)$$

$$M_{ba} = M_{Fba} + \frac{2EI}{L}\left(\theta_a - \frac{\Delta_{ab}}{L}\right) + \frac{4EI}{L}\left(\theta_b - \frac{\Delta_{ab}}{L}\right) \qquad (4.4f)$$

In this form a new variable ϕ_a and ϕ_b can be introduced where

$$\phi_a = \theta_a - \frac{\Delta_{ab}}{L} \qquad \text{and} \qquad \phi_b = \theta_b - \frac{\Delta_{ab}}{L}$$

This transformation is historically interesting since without the fixed-end moment it provides the original form of the slope-deflection equations that were derived by H. Manderla in 1879. The present form as given in equations 4.4*c*, *d* is primarily due to the development by Professor Maney of the fixed-end moments M_{Fab} and M_{Fba} to allow for the effect of the transverse forces. This important addition to the equations extended their application into the more general field of continuous frame structures. References for further study of this subject are at the end of the chapter.

4.4 *Fixed-End Couples*

By means of equations 4.7*a*, *b* the fixed-end couples that must be applied at the ends of a straight prismatic member to prevent rotation of the end cross sections (or tangents) may be determined when the angles α_a and α_b are known. In Example 3.3 the end rotations for beam *ab* are

$$\alpha_a = + \frac{200{,}000}{3EI}$$

$$\alpha_b = - \frac{250{,}000}{3EI}$$

Therefore the fixed-end couples necessary to prevent rotation are

$$M_{Fab} = - \frac{EI}{15}\left[4\left(\frac{200{,}000}{3EI}\right) + 2\left(-\frac{250{,}000}{3EI}\right)\right] = -6667 \text{ ft-lb}$$

$$M_{Fba} = - \frac{EI}{15}\left[2\left(\frac{200{,}000}{3EI}\right) + 4\left(-\frac{250{,}000}{3EI}\right)\right] = +13{,}333 \text{ ft-lb}$$

These signs for the end couples indicate that M_{Fab} is applied counterclockwise on the member and M_{Fba} is clockwise.

By this procedure the fixed-end couples may be determined and

tabulated for many standard loading conditions. For a uniform load w applied over the entire span L the end rotations are

$$\alpha_a = +\frac{wL^3}{24EI}; \qquad \alpha_b = -\frac{wL^3}{24EI}$$

and, consequently,

$$M_{Fab} = -\frac{EI}{L}\left[4\left(+\frac{wL^3}{24EI}\right) + 2\left(-\frac{wL^3}{24EI}\right)\right] = -\frac{wL^2}{12}$$

$$M_{Fba} = -\frac{EI}{L}\left[2\left(+\frac{wL^3}{24EI}\right) + 4\left(-\frac{wL^3}{24EI}\right)\right] = +\frac{wL^2}{12}$$

The fixed-end couples for various loading conditions are tabulated in Fig. 4.6. Most loading arrangements can be resolved into components for which the fixed-end couples are tabulated.

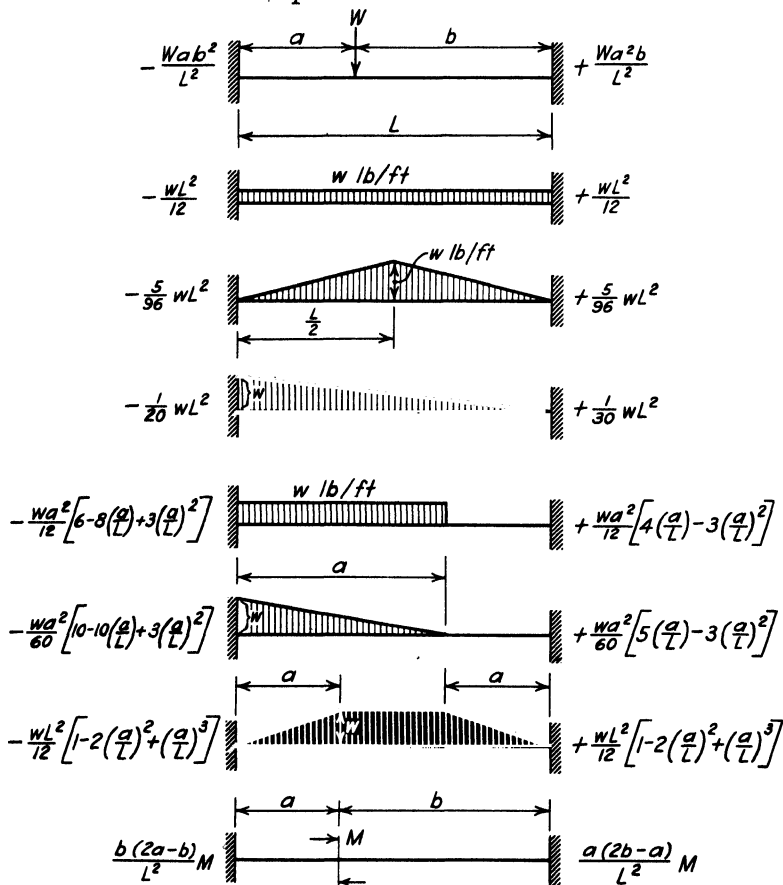



Fig. 4.6

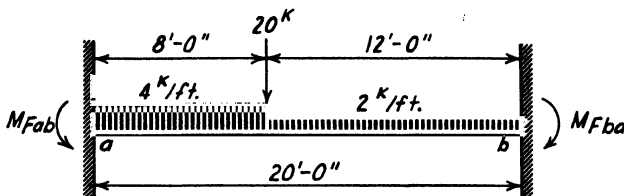



Fig. 4.7

Example 4.1 Calculate the fixed-end couples for beam ab (Fig. 4.7) from the values tabulated in Fig. 4.6.

For a uniform load of 2 kips per foot over the entire span the fixed-end couples in foot-kips are

$$M_{Fab} = -\frac{(2)(20)(20)}{12} = -66.7$$

$$M_{Fba} = +66.7$$

A uniform load of 2 kips per foot over a distance of 8 ft from end a for which a/L equals 0.4 gives fixed-end couples of

$$M_{Fab} = -\frac{(2)(8)^2}{12}(6 - 3.20 + 0.48) = -35.0$$

$$M_{Fba} = +\frac{(2)(8)^2}{12}(1.60 - 0.48) = +12.0$$

A concentrated load of 20 kips applied at 8 ft from a gives

$$M_{Fab} = -\frac{(20)(8)(12)(12)}{(20)(20)} = -57.6$$

$$M_{Fba} = +\frac{(20)(8)(8)(12)}{(20)(20)} = +38.4$$

The resultant fixed-end couples for the entire load are therefore

$$M_{Fab} = -66.7 + 35.0 + 57.6 = -159.3 \text{ ft-kips}$$

$$M_{Fba} = +66.7 + 12.0 + 38.4 = +117.1 \text{ ft-kips}$$

Example 4.2 Determine the fixed-end couple at b if the beam in Example 4.1 is simply supported at a, that is, if M_{ab} equals zero.

This condition can be obtained by adding the force system shown in Fig. 4.5a if M_{ab}' equals minus M_{Fab} and M_{ba}' equals minus $M_{Fab}/2$ or

$$M'_{Fba} = +117.1 - (\tfrac{1}{2})(-159.3) = +196.7$$

Similarly, the fixed-end couple at a when end b is simply supported is

$$M'_{Fab} = -159.3 - (\tfrac{1}{2})(+117.1) = -217.9 \text{ ft-kips}$$

From these examples it can be seen that the couple at a fixed end of a beam subjected to vertical loads increases as the couple at the other end decreases.

4.5 Summary of Slope Deflection Equations for Members with Constant EI

From the derivations and explanations that have been given in the preceding articles the relationship between the redundant end couples M_{ab} and M_{ba} and the end rotations and transverse displacements θ_a, θ_b, Δ_a, Δ_b (see Fig. 4.4*d*) for members with constant EI can be expressed in the form

$$M_{ab} = M_{Fab} + K_{ab}\left(4\theta_a + 2\theta_b + 6\,\frac{\Delta_a - \Delta_b}{L}\right) \qquad (4.8a)$$

$$M_{ba} = M_{Fba} + K_{ab}\left(2\theta_a + 4\theta_b + 6\,\frac{\Delta_a - \Delta_b}{L}\right) \qquad (4.8b)$$

In these equations the value of the coefficient K is

$$K = \frac{EI}{L} \qquad (4.9a)$$

although in most calculations where E is a constant for all members and where no volumetric changes are involved, the value of E is assumed to be unity and the value of K then becomes

$$K = \frac{I}{L} \qquad (4.9b)$$

For this assumption the calculated end rotations and displacements are, of course, equal to $E\theta_a$, $E\theta_b$, $E\Delta_a$, $E\Delta_b$.

In the development of equations 4.8*a*, *b* it was assumed that M_{ab} and M_{ba} were statically indeterminate quantities. If then any end couple is a statically determinate quantity, the equations can be modified for this condition. Thus if M_{ba} is known, from equation 4.8*b* the value of θ_b is

$$\theta_b = \frac{M_{ba} - M_{Fba}}{4K_{ab}} - \frac{\theta_a}{2} - \left(\frac{3}{2}\right)\frac{\Delta_a - \Delta_b}{L} \qquad (4.10)$$

and if this value of θ_b is substituted in equation 4.8*a*, the redundant end couple M_{ab} is expressed in the form

$$M_{ab} = M_{Fab} - \frac{M_{Fba}}{2} + \frac{M_{ba}}{2} + 3K_{ab}\theta_a + 3K_{ab}\left(\frac{\Delta_a - \Delta_b}{L}\right) \qquad (4.11a)$$

Similarly, if M_{ab} is a statically determinate quantity, then by eliminating θ_a from equation 4.8*b* the value of the redundant couple M_{ba} is

$$M_{ba} = M_{Fba} - \frac{M_{Fab}}{2} + \frac{M_{ab}}{2} + 3K_{ab}\theta_b + 3K_{ab}\left(\frac{\Delta_a - \Delta_b}{L}\right) \quad (4.11b)$$

By means of equations 4.8*a*, *b* and 4.11*a*, *b* the student should learn to write quickly and accurately the algebraic relations between the redundant end couples, end rotations, and end displacements for members with constant EI and with various end conditions. It is assumed that no discontinuities in the elastic curves of the various members are involved. Most numerical values of M_{Fab} and M_{Fba} for both magnitude and sign can be obtained from Fig. 4.6. It is again emphasized that, in the derivation of the equations, all unknowns have been assumed with positive values and therefore the proper sign must be used with any known quantity.

4.6 Equilibrium Conditions

After the unknown end moments that act upon the various members have been expressed in terms of the rotations and translations of the joints, it is necessary to consider the equilibrium requirements of the various parts of the structure. In general, the equilibrium conditions of each member and the structure as a whole must be satisfied and, further, the resultant of the external and internal forces acting on any joint must be zero. It is therefore important that both correct and sufficient conditions of equilibrium be established.

To illustrate this statement, let us consider the end forces that act on the members of the frame shown in Fig. 4.8 and their relation to the equilibrium of the structure. All end moments are assumed positive or clockwise, whereas the direction of the H and V forces can be assumed in any convenient direction if the signs are consistent throughout. The conditions of equilibrium then require that the following equations are satisfied:

$$\begin{array}{llr} (a)\ M_{ba} + M_{bc} = 0 & (d)\ H_{ab} + H_{dc} = 0 & \\ (b)\ M_{cb} + M_{cd} = 0 & (e)\ V_{ab} - V_{ba} = 0 & (4.12) \\ (c)\ V_{ab} + V_{dc} - W = 0 & (f)\ V_{dc} - V_{cd} = 0 & \end{array}$$

A study of the forces acting on each member will show that any consistent set of end moments that will satisfy equations 4.12*a*, *b*, and *d* can be used in the solution of equations 4.12*c*, *e*, and *f*. Therefore the magnitude and direction of the end moments should be calculated from equations 4.12*a*, *b*, and *d*. The first two equations, 4.12*a* and *b*, are expressed

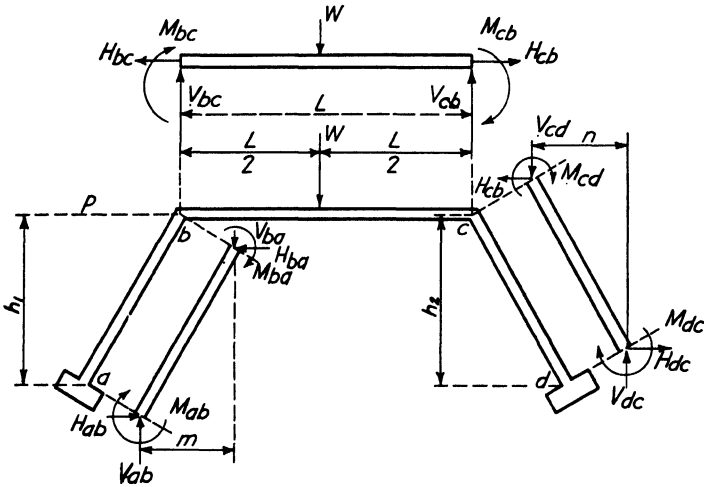



Fig. 4.8

directly in terms of the end moments, and equation 4.12d can be transformed into the same quantities by the following relations.

$$
\begin{aligned}
V_{ab} &= V_{bc} = \frac{W}{2} - \frac{M_{bc} + M_{cb}}{L} \\
H_{ab} &= \frac{M_{ab} + M_{ba} + V_{ab}m}{h_1} \\
V_{cb} &= V_{dc} = \frac{W}{2} + \frac{M_{bc} + M_{cb}}{L} \\
H_{dc} &= \frac{M_{cd} + M_{dc} - V_{dc}n}{h_2}
\end{aligned}
\tag{4.13}
$$

When these quantities are substituted in equation 4.12a, b, d, there will be three independent equations, each in terms of the unknown end

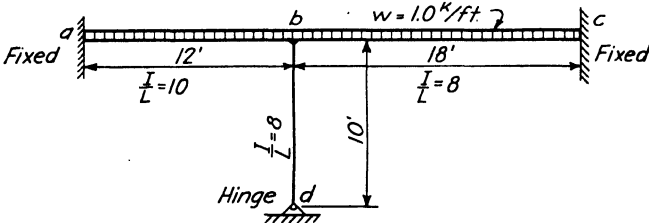



Fig. 4.9

moments. Since all end moments can be expressed in terms of θ_b θ_c, and Δ, the rotations and horizontal displacement of the joints b and c, by equations 4.8 or 4.11, there are sufficient equations for calculating these displacements. The use of the slope deflection equations in determining the end moments in certain types of rigid-frame structures will now be illustrated by several numerical examples.

Example 4.3 A structure formed by three members whose axes are represented by ab, bc, and bd (Fig. 4.9) is so constructed that the end tangents at a and c are fixed in position ($\theta_a = \theta_c = 0$), whereas the tangent at d is free to rotate (hinged at d). The members are rigidly connected at b, that is, the tangent to each member at the joint b rotates through the same angle θ_b, thus maintaining a 90° angle between the tangents. No translation is considered.

The end moments acting on each member can be expressed in terms of the rotation of joint b only by equations 4.8 or 4.11. For convenience in writing the equations, let θ_b equal E times the true angular rotation of b. Then, from equation 4.8,

$$M_{ab} = (2)(10)\theta_b + M_{Fab}$$

However,

$$M_{Fab} = -\frac{wL^2}{12} = \frac{(1)\text{ kip }(12)^2}{12} = -12\text{ ft-kips} \qquad \text{(counterclockwise)}$$

Therefore

$$\begin{aligned} M_{ab} &= 20\theta_b - 12 \\ M_{ba} &= (4)(10)\theta_b + M_{Fba} = 40\theta_b + 12.0 \\ M_{bc} &= (4)(8)\theta_b + M_{Fbc} = 32\theta_b - 27.0 \\ M_{cb} &= (2)(8)\theta_b + M_{Fcb} = 16\theta_b + 27.0 \end{aligned} \tag{4.14}$$

By equation 4.11, $M_{bd} = (3)(8)\theta_b = 24\theta_b$.

The equations are dependent on continuity of the members at joint b. Moreover, these moments must also satisfy the equilibrium condition for joint b that is,

$$M_{ba} + M_{bc} + M_{bd} = 0 \tag{4.15}$$

If the values in equation 4.14 are substituted in equation 4.15 we obtain

$$40\theta_b + 12.0 + 32\theta_b - 27.0 + 24\theta_b = 0$$

$$96\theta_b = 15.0$$

$$\theta_b = 0.1563 \qquad (\text{actually } E\theta_b)$$

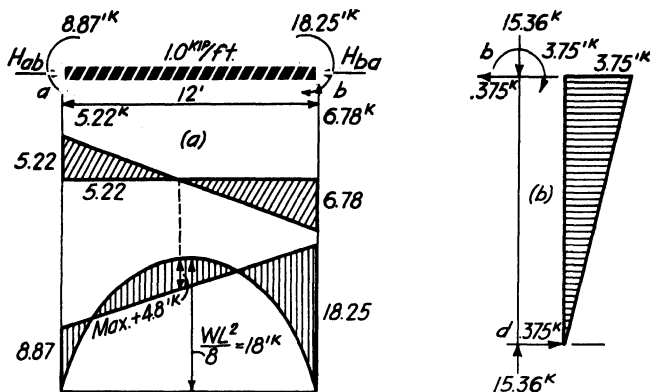



Fig. 4.10

When this value of θ_b is substituted back in equation 4.14 for the end couples, the following values are obtained:

$$M_{ab} = -8.87 \text{ ft-kips} \qquad M_{cb} = 29.5 \text{ ft-kips}$$

$$M_{ba} = 18.25 \text{ ft-kips} \qquad M_{bd} = 3.75 \text{ ft-kips}$$

$$M_{bc} = -22.00 \text{ ft-kips}$$

After the numerical values and directions of the end couples are known, the end shears for each member can be calculated and the ordinary shear and bending-moment diagrams drawn. In drawing the bending-moment diagrams, it is essential in design work to use the ordinary conventional signs for the bending moments, which are in terms of the curvature of the member. Since the analysis by means of the slope deflection equations gives the direction of the end couples acting on each member, the direction of the curvature of the axis of the member is known and consequently the usual bending-moment diagram is easily drawn. Thus for the frame in Fig. 4.9 we would obtain the shear and bending-moment diagrams in Fig. 4.10. Only the directions of the end couples (not the signs) are transferred to the diagrams in Fig. 4.10 for the calculation of shears and bending moments.

Example 4.4 The frame shown in Fig. 4.11 represents a structure which has neither angular nor linear displacement at support a, but is free to rotate at d. The members are rigidly connected at joints b and c, which have both rotation and horizontal displacement. Vertical motion of joints b and c produced by change in length of the members is neglected. The end moments can be expressed in terms of the displacement of the joints by means of equations 4.8 and 4.11. It should be noted that if all

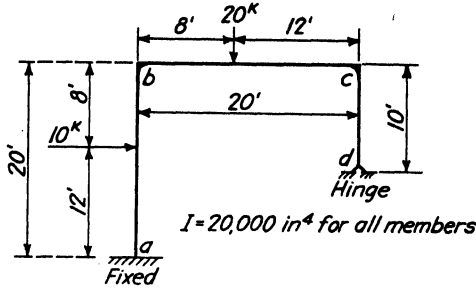



Fig. 4.11

lengths are taken in foot units, but E and I are kept in inch units, then the relative values of θ and Δ that are obtained must be multiplied by $144/E$ to obtain absolute values in radians and foot units respectively. Relative values are more convenient to use than absolute values when only the end moments are desired.

$$M_{ab} = 2\left(\frac{20{,}000}{20}\right)\theta_b - 6\left(\frac{20{,}000}{20}\right)\frac{\Delta}{20} - \frac{(10)(8)(12)(8)}{(20)(20)}$$

or

$$M_{ab} = 2000\theta_b - 300\Delta - 19.2$$
$$M_{ba} = 4000\theta_b - 300\Delta + 28.8$$
$$M_{bc} = 4000\theta_b + 2000\theta_c - 57.6 \qquad (4.16)$$
$$M_{cb} = 2000\theta_b + 4000\theta_c + 38.4$$

$$M_{cd} = 3\left(\frac{20{,}000}{10}\right)\theta_c - 3\left(\frac{20{,}000}{10}\right)\frac{\Delta}{10}$$

or

$$M_{cb} = 6000\theta_c - 600\Delta$$

The displacement Δ is assumed to the right and will therefore produce negative moments. However, it could just as well have been assumed to the left, that is, $+6K(\Delta/L)$ could have been used. Then Δ would have come out negative.

The end moments must satisfy the following equilibrium equations:

$$M_{ba} + M_{bc} = 0 \qquad (4.17a)$$

$$M_{cb} + M_{cd} = 0 \qquad (4.17b)$$

$$H_a + H_d + 10 = 0 \qquad (4.17c)$$

Equations 4.17*a* and 4.17*b* are expressed directly in terms of the end moments so that only equation 4.17*c* need be modified. By considering

the equilibrium of members *ab* and *cd* for the forces shown in Fig. 4.12, we can write

$$H_a = \frac{M_{ab} + M_{ba} - 80}{20}$$

$$H_d = \frac{M_{cd}}{10}$$

By substituting the values of H_a and H_d in equation 4.17*c*, this equation is then expressed in terms of the end moments.

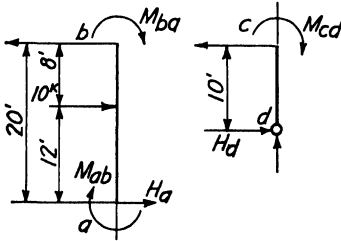



Fig. 4.12

$$\frac{M_{ab} + M_{ba} - 80}{20} + \frac{M_{cd}}{10} + 10 = 0 \qquad (4.17d)$$

or

$$M_{ab} + M_{ba} + 2M_{cd} = -120 \qquad (4.17e)$$

When the values of the moments in equations 4.17*a*, *b*, and *e* are expressed in terms of the displacements of the joints by equations 4.16, the following equations are obtained:

$$\begin{aligned} 8000\theta_b + 2000\theta_c - 300\Delta &= 28.8 \\ 2000\theta_b + 10{,}000\theta_c - 600\Delta &= -38.4 \\ 6000\theta_b + 12{,}000\theta_c - 1800\Delta &= -129.6 \end{aligned} \qquad (4.18)$$

Solving these equations, we obtain

$$\theta_b = 0.0072$$

$$\theta_c = 0.0008$$

$$\Delta = 0.1013$$

Substituting these values in equations 4.16 gives

$$M_{ab} = -35.2 \text{ ft-kips} \qquad M_{cb} = 56.0 \text{ ft-kips}$$

$$M_{ba} = 27.2 \text{ ft-kips} \qquad M_{cd} = -56.0 \text{ ft-kips}$$

$$M_{bc} = -27.2 \text{ ft-kips}$$

Example 4.5 The frame in Fig. 4.13 illustrates a problem involving a cantilever member and an eccentric load on a column, two conditions that are often encountered. The fixed-end moment M'_{Fcd} produced by the eccentric load of 6 kips can be calculated by the procedure used in

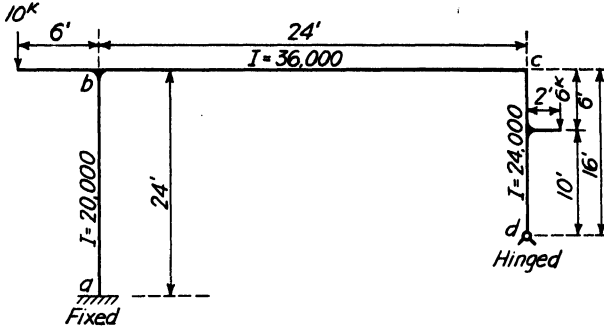



Fig. 4.13

Example 4.2. From Fig. 4.6 the end couples at c and d for an applied moment of 12 ft-kips, when both ends c and d are fixed, are

$$M_{Fcd} = \frac{(10)(12 - 10)(12)}{(16)(16)} = 0.938 \text{ ft-kips}$$

$$M_{Fdc} = \frac{(6)(20 - 6)(12)}{(16)(16)} = 3.936 \text{ ft-kips}$$

When the moment at end d is zero, the fixed-end couple M'_{Fcd} at c is

$$M'_{Fcd} = 0.938 - (\tfrac{1}{2})(3.936) = -1.03 \text{ ft-kips}$$

The slope deflection equations for all end moments in Fig. 4.13 can now be written in the following form if the horizontal displacement Δ of joints b and c is assumed to the left.

$$M_{ab} = 2\left(\frac{20{,}000}{24}\right)\theta_b + 6\left(\frac{20{,}000}{24}\right)\frac{\Delta}{24} = 1667\theta_b + 208.3\Delta$$

$$M_{ba} = 3334\theta_b + 208.3\Delta$$

$$M_{bc} = \left(\frac{36{,}000}{24}\right)(4\theta_b + 2\theta_c) = 6000\theta_b + 3000\theta_c \tag{4.19}$$

$$M_{cb} = 3000\theta_b + 6000\theta_c$$

$$M_{cd} = \left(\frac{24{,}000}{16}\right)\left(3\theta_c + \frac{3\Delta}{16}\right) - 1.03 = 4500\theta_c + 281.3\Delta - 1.03$$

The end moments must satisfy the following equilibrium conditions:

$$M_{ba} + M_{bc} + 60 = 0 \tag{4.20a}$$

$$M_{cb} + M_{cd} = 0 \tag{4.20b}$$

$$H_a + H_d = 0 \tag{4.20c}$$

Equation 4.20c can be written

$$\frac{M_{ab} + M_{ba}}{24} + \frac{M_{cd} + 12}{16} = 0 \tag{4.20d}$$

Substituting the value of the moments from equation 4.19 in equations 4.20a, b, and d gives

$$9334\theta_b + 3000\theta_c + 208.3\Delta = -60 \tag{4.21a}$$

$$3000\theta_b + 10{,}500\theta_c + 281.3\Delta = 1.03 \tag{4.21b}$$

$$5000\theta_b + 6750\theta_c + 838.6\Delta = -16.46 \tag{4.21c}$$

A solution of these equations gives

$$\theta_b = -0.00725 \qquad \theta_c = 0.00196 \qquad \Delta = 0.00783$$

and when these values are substituted in equation 4.19, we obtain the following moments in foot-kips.

$$M_{ab} = -10.5 \qquad M_{cb} = -10.0$$

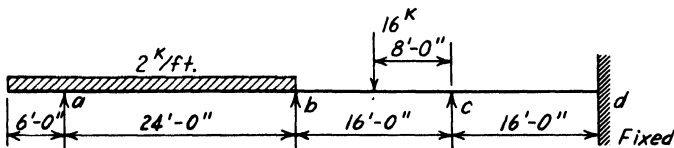
$$M_{ba} = -22.6 \qquad M_{cd} = +10.0$$

$$M_{bc} = -37.6$$

Problems

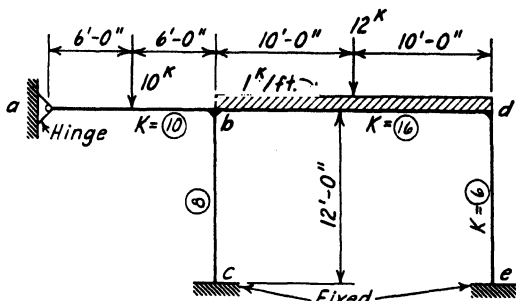
4.1 (*a*) Calculate all end moments for the continuous beam shown by the slope deflection method. Assume that EI is constant for all spans.

(*b*) Determine the maximum positive bending moment in span *ab*.

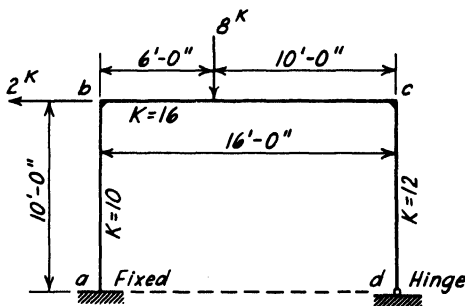
(*c*) Indicate the shape of the elastic curve by a colored line.

Problem 4.1

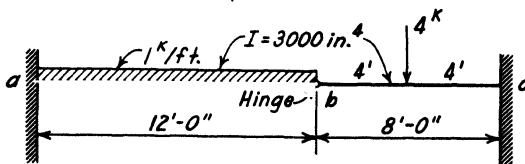
4.2 (*a*) Compute the end moments for the frame shown by means of the slope deflection equations.

(*b*) Draw the shear and bending-moment diagrams for all members and give the value of the controlling ordinates.

(*c*) Show the deformed position of the frame by a colored line.

Problem 4.2

4.3 Determine all end moments for the frame in Problem 4.2 if end a has a support which permits horizontal motion and rotation but no vertical displacement.

4.4 Determine the end moments by the slope deflection method and show the elastic curve of the frame by a colored line.

Problem 4.4

4.5 If the member abc has restrained supports (not fixed) at a and c and a movable hinge at b, express the moments at a and c in terms of the rotations θ_a and θ_c only. Keep lengths in foot units. (*Suggestion:* Write the slope deflection equations for moments M_{ab} and M_{cb} and then eliminate Δ_b by satisfying the shear condition at b.)

Problem 4.5

4.6 Determine the end moments for all members of the frame in Problem 2.25.

4.7 Solve Problems 3.29 and 3.30 by the slope deflection equations.

MOMENT-DISTRIBUTION OR CROSS METHOD

4.7 Moment-Distribution Method

A method of successive approximations that is especially useful in the analysis of continuous frame structures was presented in 1932 by Professor Hardy Cross (*Trans. Am. Soc. C. E.*, Vol. 96). This method, which is commonly referred to as the moment-distribution or Cross method, can be applied directly to continuous structures that are acted on by force systems that will prevent any translation of the joints. In other words, the frame must be supported so that the fixed-end moments in equations 4.8 and 4.11 are modified only by the rotations θ of the joints. In calculating the corrections that must be added to the fixed-end moments because of the rotation of the joints, it is possible to carry out the numerical operations by recording the change in the moments produced by the successive rotation of each joint. The entire frame can therefore be analyzed if a convenient method for determining the end moments due to the rotation of one joint is known, for that operation can be repeated until all the necessary strain and equilibrium conditions are satisfied.

Let us consider the frame shown in Fig. 4.14, in which joint x (any joint) is allowed to rotate until it is in equilibrium, whereas ends a, b, and c are fixed and d is hinged. As no translation during the rotation is permitted, any change in length of the members must be neglected or corrected later.

The moments at the ends of the members meeting at the joint x can be expressed in the usual form by equations 4.8 and 4.11.

$$\begin{aligned} M_{xa} &= M_{Fxa} + 4K_1\theta_x \qquad & M_{xb} &= M_{Fxb} + 4K_2\theta_x \\ M_{xc} &= M_{Fxc} + 4K_3\theta_x \qquad & M_{xd} &= M'_{Fxd} + 3K_4\theta_x \end{aligned} \tag{4.22}$$

in which

$$M'_{Fxd} = M_{Fxd} - \tfrac{1}{2}M_{Fdx}$$

For equilibrium

$$M_{xa} + M_{xb} + M_{xc} + M_{xd} = 0$$

Substituting the value of the moments in the equilibrium equation gives

$$(4K_1 + 4K_2 + 4K_3 + 3K_4)\theta_x + (M_{Fxa} + M_{Fxb} + M_{Fxc} + M'_{Fxd}) = 0$$

Let

$$\sum CK = 4K_1 + 4K_2 + 4K_3 + 3K_4$$

where C equals 3 or 4, depending on whether the opposite end of the member is hinged or fixed.

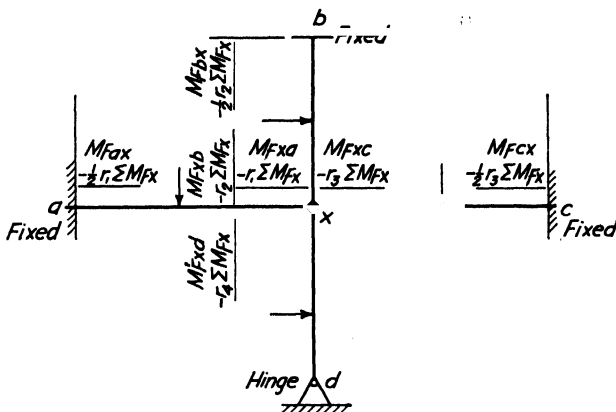



Fig. 4.14 Arrangement of values for moment-distribution method.

If

$$\sum M_{Fx} = M_{Fxa} + M_{Fxb} + M_{Fxc} + M'_{Fxd}$$

then

$$\theta_x = -\frac{\sum M_{Fx}}{\sum CK} \tag{4.23}$$

and the value of the moments in equation 4.22 will therefore be

$$\begin{aligned}
M_{xa} &= M_{Fxa} + \frac{4K_1}{\sum CK}(-\sum M_{Fx}) = M_{Fxa} + r_1(-\sum M_{Fx}) \\
M_{xb} &= M_{Fxb} + \frac{4K_2}{\sum CK}(-\sum M_{Fx}) = M_{Fxb} + r_2(-\sum M_{Fx}) \\
M_{xc} &= M_{Fxc} + \frac{4K_3}{\sum CK}(-\sum M_{Fx}) = M_{Fxc} + r_3(-\sum M_{Fx}) \\
M_{xd} &= M'_{Fxd} + \frac{3K_4}{\sum CK}(-\sum M_{Fx}) = M'_{Fxd} + r_4(-\sum M_{Fx})
\end{aligned} \tag{4.24}$$

These equations can be stated in the following way. The correction that must be added to the fixed-end moment acting on any member at a particular joint, because of the rotation of that joint, is equal to the ratio $r = CK/\Sigma\, CK$ times minus the algebraic sum of all the fixed-end moments at the joint. It should be always kept in mind that we are considering the end moments acting on the members and that clockwise moments are taken positive.

If the other end of the member has been held fixed during the rotation of joint x, the fixed-end moment at that end has also been changed by

the rotation θ_x, as expressed by the equation

$$M_{ax} = M_{Fax} + 2K_1\theta_x = M_{Fax} + \tfrac{1}{2}r_1(-\sum M_{Fx}) \qquad (4.25)$$

The correction at the end a due to the rotation θ_x is therefore one-half the correction at the x end (see Fig. 4.5b). The corrections at b and c are also one-half the corresponding corrections at x for members bx and cx. No moment is caused at point d, for the correction at x for the member xd was calculated for a hinge condition at d. In other words, if a member has a hinged end, no moments need be considered at that end because of the rotation of any joint.

Since the various joints can be rotated separately and the moments produced by the rotation recorded as shown in Fig. 4.14, the true value of the end moment is approached as the correction $r(-\Sigma\ M_{Fx})$ approaches zero. The numerical procedure is most easily explained by specific examples.

Example 4.6 The frame shown in Figs. 4.9 and 4.15 can be solved directly by equations 4.24 and 4.25 for only the rotation at joint b affects the fixed-end moments. Particular attention should be given to the arrangement in Fig. 4.15 for computing the distribution factors r, as this procedure is especially desirable when members with variable moments of inertia are used. The algebraic sum of the fixed-end moments at joint b equals $-27.0 + 12.0$ or -15.0. Therefore a moment of r (distribution factor) times $+15.0$ is added to each member at joint b. Thus the correction for M_{ba} equals (0.417)(15), or 6.25, and the correction added to M_{ab} is half of this value, or 3.12. Notice that both corrections have the same sign.

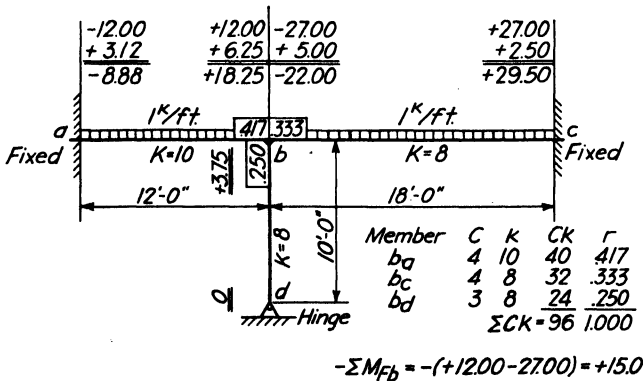



Fig. 4.15

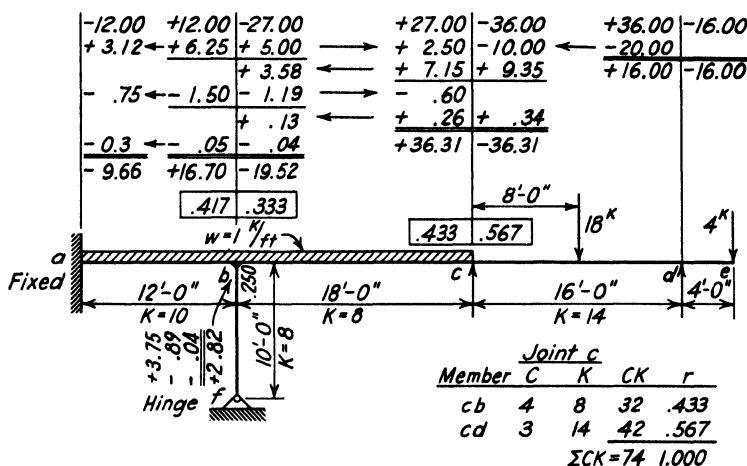



Fig. 4.16

Example 4.7 When the equilibrium and strain conditions at more than one joint must be considered, the numerical procedure used in Example 4.6 must be repeated until the corrections may be ignored. Thus the total rotation of any joint is obtained from equation 4.23 as the summation of a series of rotations, each of which must satisfy the equilibrium conditions. The use of the moment-distribution method will be illustrated by the calculation of the end moments for the members of the continuous frame in Fig. 4.16. The solution involves the following numerical operations.

1. The calculation of the fixed-end couples for spans *ab* and *bc* which are the same as for Example 4.6 is made in the same manner. For span *cd*,

$$M_{Fcd} = -\frac{(18)(16)}{8} = -36.0$$

$$M_{Fdc} = +36.0$$

However, since M_{dc} is statically determinate with a value of +16.0, from equation 4.11*a*

$$M'_{Fcd} = M_{Fcd} - \frac{M_{Fdc} - M_{dc}}{2} = -36.0 - \frac{+36.0 - 16.0}{2} = -46.0$$

The arrangement in Fig. 4.16 shows these values, −36.0 and −10.0, recorded separately. In all succeeding calculations the moment M_{dc} will be kept constant, that is, the change in moment because of any joint rotation is zero.

2. The calculation of the distribution factors for joint b is given in Example 4.6 and for joint c in Fig. 4.16. These distribution factors represent the following boundary conditions: when joint b rotates, ends a and c have no rotation and f has no change in moment, and when joint c rotates, end b has no rotation and end d has no change in moment. Any end that has no rotation must have restraining moments, and any end that has no change in moment must have rotation. Note the use of C values of 4 and 3 to allow for these different boundary conditions.
3. The corrections of the end moments because of rotation of joint b were recorded first. These corrections are the same as in Example 4.6. Then joint c is permitted to rotate to a condition of equilibrium by correcting for the total unbalanced fixed-end couples of

$$M_{Fc} = +27.00 + 2.50 - 36.00 - 10.00 = -16.50$$

The corrections are therefore

$$\Delta M_{cb} = (0.433)[-(-16.5)] = +7.15$$
$$\Delta M_{cd} = (0.567)[-(-16.5)] = +9.35$$

Because joint b is held against rotation ($C = 4$), an additional fixed-end couple of $+7.15/2$ must be applied, but no change in moment at d is necessary ($C = 3$).
4. Since joint b now has an unbalanced fixed-end couple of 3.58, it must again rotate to a condition of equilibrium which requires corrections of

$$M_{ba} = (0.417)(-3.58) = -1.50$$
$$M_{bc} = (0.333)(-3.58) = -1.19$$
$$M_{bf} = (0.250)(-3.58) = -0.89$$

All carry-over operations are indicated by arrows. The entire solution requires three rotations of joint b and two rotations of joint c.
5. When all corrections have been made, a double line should be drawn and the total value of the end couples recorded.

Example 4.8 The moment-distribution method will also be used to analyze the frame shown in Figs. 4.11 and 4.17. Because only the change in end moments due to the rotation of the joints can be considered in the moment-distribution method, an auxiliary force F must be applied, as shown in Fig. 4.17, to prevent any horizontal movement of b and c. The fixed-end moments produced by the loading are then corrected for the successive rotation of the joints b and c. Joint c is balanced first, and

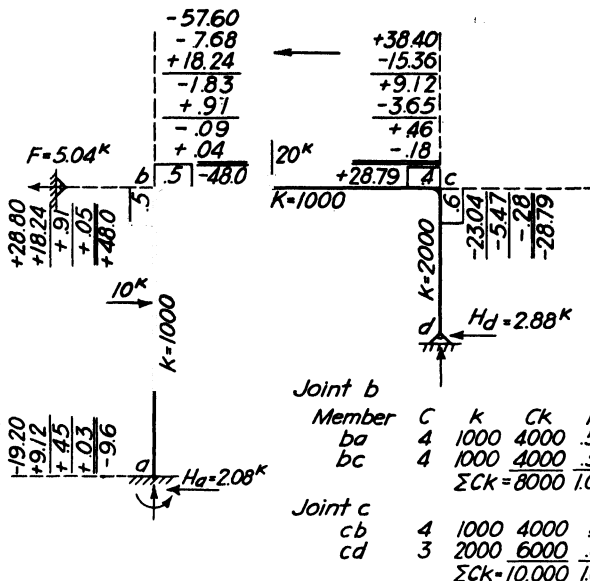



Fig. 4.17

then joint b, the various operations being indicated in Fig. 4.17. After the end moments are determined, the value of the auxiliary force F is calculated from the equilibrium condition

$$H_a + H_d + F - 10 = 0$$

or

$$F = 5.04 \text{ kips}$$

The force F must now be removed from the force system by applying an equal and opposite force.

To determine the moments caused by a force equal but opposite to F, the following procedure is used. Assume that the joints b and c are moved horizontally to the right some arbitrary amount Δ' by a force F' (see Fig. 4.18a). If no rotation of the joints a, b, and c is to take place, a fixed-end moment of $6K_{ab}(E\Delta'/L)$ is required at each end of the member ab and a moment of $3K_{cd}(E\Delta'/L)$ at joint c for member cd. (See equations 4.8 and 4.11.) These fixed-end moments are corrected for the rotation of the joints in the same manner as were the fixed-end moments for the transverse loading in Fig. 4.17. Thus, if an assumed displacement of $E\Delta' = 0.10$ is taken to the right, the fixed-end moments at both ends of ab are

$$M_{Fab} = M_{Fba} = -\frac{(6)(1000)(0.10)}{20} = -30$$

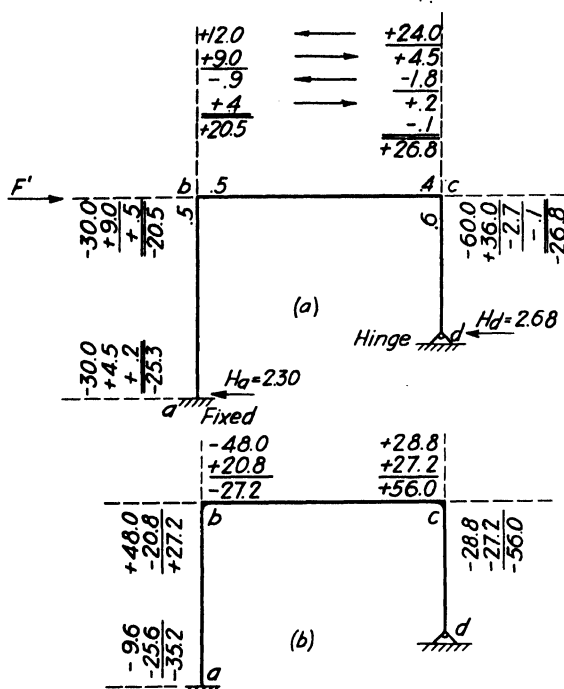



Fig. 4.18

and at joint c for member cd

$$M_{Fcd} = -\frac{(3)(2000)(0.10)}{10} = -60$$

These fixed-end moments are corrected for rotation of the joints as shown in Fig. 4.18a. If the actual value of Δ is to be used, it must be kept in the same units as E, I, and L; but if only the moments that will provide equilibrium are required, any choice of units can be made, provided that such use is consistent throughout the calculations.

The value of the force F' necessary to give the displacement $E\Delta'$ equal to 0.10 and the moments recorded in Fig. 4.18a is equal to

$$F' = H_a + H_d = 2.30 + 2.68 = 4.98$$

However, a force $-F$ equal to 5.04 kips was required, which by Hooke's law can be obtained by increasing the value of F' and the corresponding moments by

$$\frac{F}{F'} = \frac{5.04}{4.98}$$

Consequently, the true value of the moment $M_{ab}(+35.2)$ is the algebraic sum of -9.6 ft-kips, the moment caused by the actual force system including F, and

$$\frac{5.04}{4.98}(-25.3) = -25.6$$

which is the moment due to $-F$. The final values of the moments are recorded in Fig. 4.18*b*.

4.8 *Variation in the Value of C*

In the preceding discussion of the relation between an end couple and the corresponding rotation, the opposite end of the member was assumed to be either fixed or hinged. However, at various times it will be convenient to consider other boundary conditions at the opposite end. For instance, in Fig. 4.19*a* a prismatic member is shown which has equal and opposite rotations at the two ends, that is,

$$\theta_b = -\theta_a, \qquad M_{ba} = -M_{ab}$$

For this case equation 4.2*a* gives

$$M_{ab} = 4\frac{EI}{L}\theta_a + 2\frac{EI}{L}(-\theta_a) = 2\frac{EI}{L}\theta_a$$

and

$$M_{ba} = 2\frac{EI}{L}\theta_a + 4\frac{EI}{L}(-\theta_a) = -2\frac{EI}{L}\theta_a \tag{4.26}$$

In other words, the C value for this case is 2 and the moments at the two ends are equal and opposite.

When the ends of a member of constant cross section have end rotations

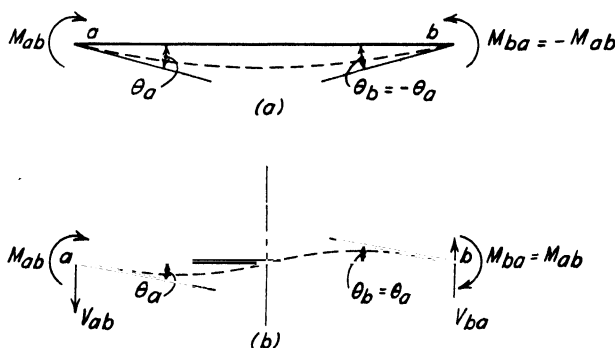



Fig. 4.19

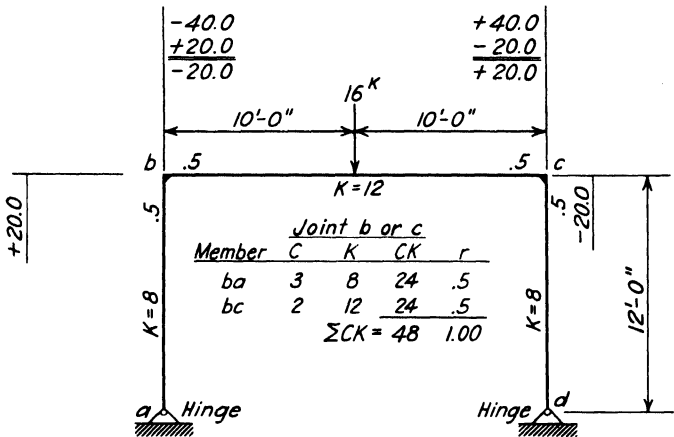



Fig. 4.20

and end couples of the same magnitude and direction as in Fig. 4.19*b*, that is,

$$\theta_b = \theta_a \qquad \text{and} \qquad M_{ba} = M_{ab}$$

equation 4.2*a* gives

$$M_{ab} = 4\frac{EI}{L}\theta_a + 2\frac{EI}{L}\theta_a = 6\frac{EI}{L}\theta_a$$

and

$$M_{ba} = 2\frac{EI}{L}\theta_a + 4\frac{EI}{L}\theta_a = 6\frac{EI}{L}\theta_a \tag{4.27}$$

Thus for this case the coefficient C is 6 and the two end couples are the same in both magnitude and direction.

Example 4.9 As the frame in Fig. 4.20 is symmetrical with respect to both physical characteristics and loading, it is apparent that the rotation at joint c is equal and opposite to that at joint b. Therefore, if a coefficient C equal to 2 (equation 4.26) is used for member bc and a value of 3 for members ab and cd, the boundary conditions throughout the structure will be satisfied with one simultaneous rotation of joints b and c as shown.

Example 4.10 The frame in Fig. 4.21 is essentially symmetrical about the center line if the change in the length of the members is neglected. In this case, joints b, b', c, and c' will have the same horizontal displacement Δ, and the rotation of joint c' will be equal and in the same direction as joint c. Therefore for member cc' the correct boundary conditions can be maintained if a coefficient C of 6 (equation 4.27) is used in determining the distribution factors at joints c and c'. Thus, for joint c, the distribution

factors r can be determined as follows:

Member	C	K	CK	r
cb	4	9	36	0.375
cd	3	4	12	0.125
cc'	6	8	48	0.500
			$\sum CK = 96$	1.000

The distribution factors for joint b are determined by using C equal to 4 for both ba and bc, which means that ends a and c are restrained against rotation when joint b is rotated to a condition of equilibrium.

The fixed end moments in the columns are determined for an arbitrary value of $E\Delta/h$ equal to unity and to the right as indicated, which gives the following values:

$$M_{Fab} = M_{Fba} = -(6)(6)(1) = -36$$

$$M_{Fcd} = -(3)(4)(1) = -12$$

Any consistent set of units can be used in these calculations.

The joint rotations are started at b as shown in Fig. 4.21. The advantage of making the corrections at both ends of member bc simultaneously is apparent as the fixed-end couple 10.8 almost balances the fixed-end couple (−12.0). The corrections due to the rotation of joint c are consequently small, and therefore only two corrections are required at joint b and one at joint c.

The final end couples and column shears, together with the applied auxiliary force F', are recorded in Fig. 4.21. For any actual applied load F the values shown must be multiplied by the ratio F/F' as illustrated in Example 4.8.

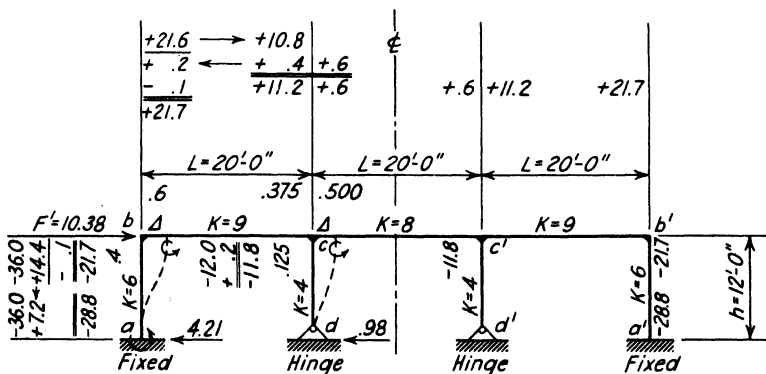



Fig. 4.21

4.9 Effect of Shearing Deformation

The deformation due to shearing stresses reduces the CK, or stiffness factor, and the carry-over factor. The fixed-end moments are also modified when the loading is unsymmetrical. The amount of the reduction is readily calculated by including the strain energy due to shear in the solution of the problems that are given in Fig. 4.5*a*, *b*, and *c*. Thus in Fig. 4.5*a* the total strain energy is

$$U = \int_0^L \frac{(M_{ab}' - V_a'x)^2\,dx}{2EI} + \int_0^L \frac{V_a'^2\,dx}{2AG} \tag{4.28}$$

By applying Castigliano's theorem, we obtain

$$\Delta_a = \frac{\partial U}{\partial V_a'} = \int_0^L \frac{(M_{ab}' - V_a'x)(-x)\,dx}{EI} + \int_0^L \frac{V_a'\,dx}{AG} = 0 \tag{4.29}$$

from which

$$V_a' = \frac{M_{ab}'}{L}\left(\frac{1.5}{1+j}\right) \tag{4.30}$$

where

$$j = \frac{3EI}{L^2AG}$$

The relationship between the end moment M_{ab}' and rotation θ_a of the end section is

$$\theta_a = \frac{\partial U}{\partial M_{ab}'} = \int_0^L \frac{(M_{ab}' - V_a'x)\,dx}{EI}$$

from which, using the value of V_a' in equation 4.30,

$$M_{ab}' = \left(\frac{1+j}{0.25+j}\right)K\theta_a \tag{4.31a}$$

and

$$M_{ba}' = \left(\frac{0.5-j}{1+j}\right)M_{ab}' \tag{4.31b}$$

If $M_{ba}' = 0$ (hinge at b),

$$M_{ab}' = \left(\frac{3}{1+j}\right)K\theta_a \tag{4.32a}$$

From equations 4.31*a* and 4.32*a*, the stiffness factors CK are

$$\left(\frac{1+j}{0.25+j}\right)K \quad \text{instead of } 4K \text{ for fixed ends} \tag{4.32b}$$

$$\left(\frac{3}{1+j}\right)K \quad \text{instead of } 3K \text{ for hinged ends} \tag{4.32c}$$

and the carry-over factor when end b is fixed is

$$\frac{0.5 - j}{1 + j} \quad \text{instead of } 0.5 \tag{4.32d}$$

For steel beams the ratio E/G is approximately 2.6, and the ratio I/A depends upon the depth of the beam and the shape of the cross section. For I-beams or girders, the area of the web should be used for A which tends to increase the value of j. The effect of shearing deformation is most important for short, deep I-sections, but even for them the change in the end moments can usually be neglected.

The fixed-end moments for symmetrical loads on the beam are not changed by shearing deformation. This is because the end sections remain vertical when the shearing displacements occur as long as the deformation is symmetrical about the center line. Therefore the rotations α of the end sections is due to bending moments only. If $\alpha_a = -\alpha_b$,

$$M_{Fab} = -\left(\frac{1 + j}{0.25 + j}\right)K\alpha_a + \left(\frac{0.5 - j}{0.25 + j}\right)K\alpha_a$$

or

$$M_{Fab} = \left(\frac{-0.5 - 2j}{0.25 + j}\right)K\alpha_a = -2K\alpha_a \tag{4.33}$$

which is the same as when the shearing force is not considered.

When transverse loads on the beam are unsymmetrical, the fixed-end moments are modified by the shearing deformation even though the rotations α_a and α_b are not. When both ends are considered fixed, the end couples can be determined from the equations

$$M_{Fab} = \left(\frac{1 + j}{0.25 + j}\right)K(-\alpha_a) + \left(\frac{0.5 - j}{0.25 + j}\right)K(-\alpha_b) \tag{4.34a}$$

$$M_{Fba} = \left(\frac{0.5 - j}{0.25 + j}\right)K(-\alpha_a) + \left(\frac{1 + j}{0.25 + j}\right)K(-\alpha_b) \tag{4.34b}$$

It is apparent that the changes in the fixed-end couples due to shearing deformation are caused by the variation in the coefficients and are not due to any changes in α_a and α_b. This statement applies to transverse forces only.

If the beam is fixed at end a and is simply supported at end b, the fixed-end moment M'_{Fab} can be determined directly from equation 4.34c.

$$M'_{Fab} = \left(\frac{3}{1 + j}\right)K(-\alpha_a) \tag{4.34c}$$

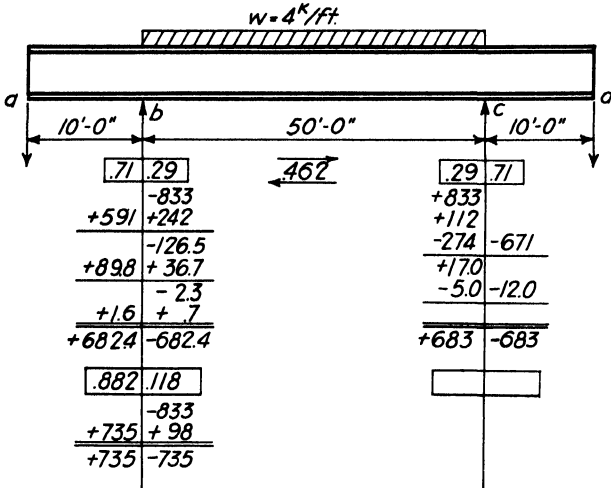



Fig. 4.22

Example 4.11 A plate girder is continuous over three spans as shown in Fig. 4.22. The moment of inertia I is 20,000 in.^{4}, and the area of the web is 16 in.^{2} A comparison is now made of the maximum and minimum moments in the girder that are obtained by first considering the shearing deformation and then by neglecting it.

The values of j and CK for the end and center spans are the following. For the end spans, assuming $E/G = 2.5$,

$$j = \frac{(3)(2.5)(20{,}000)}{(120)^2(16)} = 0.65$$

For the center span,

$$j = 0.026 \qquad \text{and} \qquad CK = \left(\frac{1 + 0.026}{0.25 + 0.026}\right)\left(\frac{20{,}000}{600}\right) = 123.8$$

For the end span,

$$CK = \left(\frac{3}{1 + 0.65}\right)\left(\frac{20{,}000}{120}\right) = 303$$

Carry-over factor for the center span is

$$\frac{0.5 - 0.026}{1 + 0.026} = 0.462$$

The distribution factors at b and c are

$$r_{ba} = \frac{303}{426.8} = 0.71 \qquad r_{bc} = \frac{123.8}{426.8} = 0.29$$

Because of symmetry the fixed-end moments are not affected by the shearing deformation and therefore are equal to

$$\frac{wL^2}{12} = \frac{(4)(50)(50)}{32} = \pm 833$$

The numerical operations shown in Fig. 4.22 follow the usual procedure and give the final bending moments at b and c practically equal to -683 ft-kips. These end moments give a positive bending moment of 567 ft-kips at the center of span bc.

If the moments are now calculated by the usual distribution and carry-over factors in which the shearing deformation is neglected, the bending moments at b and c are equal to -735 and the positive moment at the center of span bc is 515 ft-kips. Therefore the shearing deformation decreases the negative bending moments at b and c by 7% and increases the positive moment at the center of span bc by 10%. The distribution factors 0.882 and 0.118 are obtained by taking advantage of the symmetry of the structure. Since $\theta_c = -\theta_b$,

$$M_{bc} = M_{Fbc} + 4K\theta_b + 2K\theta_c = M_{Fbc} + 2K\theta_b$$

Therefore, if C equal to 2 is used for member bc, the problem is solved with one distribution, as was discussed in Example 4.9.

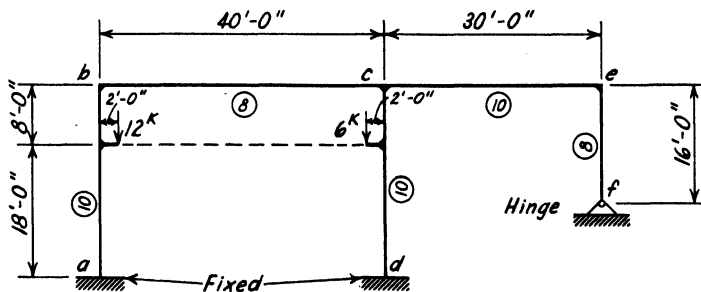
Problems

4.8 Solve Problem 4.1*a* by the moment-distribution method.

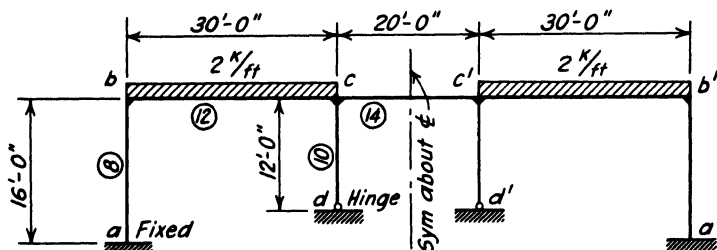
4.9 Solve Problem 4.2*a* by the moment-distribution method.

4.10 (*a*) Calculate all end moments by the moment-distribution method.

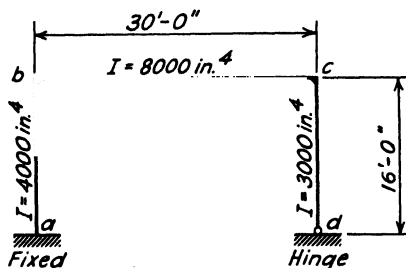
(*b*) Draw the shear and bending-moment diagrams for member *ab* and give the numerical value of each controlling ordinate.

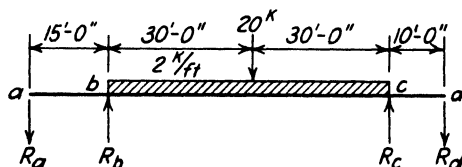
(*c*) Show the shape of the elastic curve of each member on a diagram of the frame.

Problem 4.10

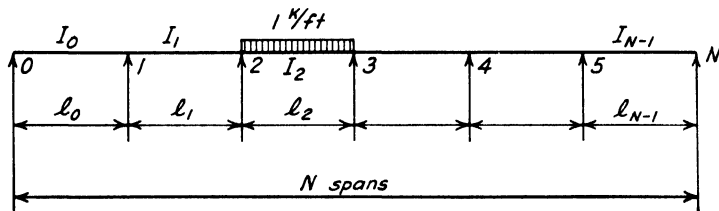
4.11 Solve Problem 4.4 by the moment-distribution method.

4.12 Calculate all end moments by the moment-distribution method using the rotation of joints b and c only.

Problem 4.12

4.13 Using the continuous frame in Problem 4.12, solve for the end moments in the members if the vertical loads are replaced by a horizontal load of 4 kips applied at joint b. Assume that the load acts toward the right and take advantage of the symmetry of the structure in your solution.

4.14 If the member bc elongates 0.15 in. as the result of a change in temperature, determine the end moments in all members by the moment-distribution method. $E = 30 \times 10^6$ lb per square inch. (*Suggestion:* Hold joint b without translation by an auxiliary force F, and then remove F as in Example 4.8.)

Problem 4.14

4.15 If the continuous girder shown has a value of I of 24,000 in.^{4} and a web area of 18 in.^{2}, determine the maximum positive and negative bending moments. Consider the effect of shear deformation in the end spans. Assume E/G equal to 2.6. Compare your results with the corresponding values when the shear deformation is neglected.

Problem 4.15

4.16 Write a computer program that will give the moments at the supports of a continuous beam of NS spans as shown in the diagram for a uniform load of 1 kip per foot on any span. Use the slope deflection equations with a solution by the iteration method.

Problem 4.16

References

1 G. A. Maney, *Engineering Studies* 1 (*Slope Deflection Method*), University of Minnesota, 1915.

2 Wilson, Richart, and Weiss, "Analysis of Statically Indeterminate Structures by the Slope-Deflection Method," *Bull.*, 108, University of Illinois.

3 Hool and Kinne, *Stresses in Framed Structures*, McGraw-Hill Book Co.

4 A. Ostenfeld, "Die deformations Methode," *Der Bauingeniur*, January 1923.

5 Hardy Cross, "Analysis of Continuous Frames by Distributing Fixed-End Moments," *Trans. Am. Soc. C.E.*, Vol. 96 (1932).

6 "Moment Distribution Applied to Continuous Concrete Structures," *Portland Cement Assoc. Bull. S.T.*, 40.

7 John I. Parcel and R. B. B. Moorman, *Analysis of Statically Indeterminate Structures*, Chapters 5 and 6, John Wiley and Sons.

5

Building Frames Subjected to Vertical Loads

5.1 Introduction

At the present time the design of reinforced concrete buildings and many types of steel structures is based primarily on a recognition of continuity among the various structural elements. The degree of continuity that can be expected in reinforced concrete buildings requires careful study, but, in general, when the floors, girders, and columns have been properly reinforced and constructed with adequate supervision, the assumptions already made for rigid-frame structures will apply. Since continuity can exist in all directions, a building is a space structure and must be so regarded if its true structural action is to be properly interpreted. This means that the floors act as slabs and, with the girders, are subjected to bending, shear, and torsion, and that the columns are subjected to direct stress, together with shear and bending in two directions. If no girders are used, but the floor slabs are supported directly on the columns, the building falls in the classification of a flat slab structure which is frequently designed by standard coefficients that have been determined from a mathematical and experimental study of slabs under various loading and boundary conditions. However, the analysis of such structures as equivalent frames is recommended by the A.C.I. Building Code, which specifies definite assumptions for making the numerical calculations.

When the floor slab is supported by beams and girders that frame into columns, the usual procedure is to neglect the torsion in the girders and to calculate as a two-dimensional problem the stresses produced by the bending in each direction. Under these conditions the frame becomes similar to the idealized structures that have already been discussed except that the number of members is greatly increased. At this time it should be emphasized that practically no indeterminate structure is actually designed as a single unit, but instead a number of substitute structures representing the action of the actual structure as closely as possible are proportioned and analyzed. The analysis of the substitute structure is considered first and then its relation to the actual conditions is discussed.

In buildings with steel framework, the amount and effect of the deformation that may occur in the connections of the beam and girders to the columns constitute a most important and difficult problem. Most riveted connections will be of a semirigid type, although some types can be made to undergo very small deformations under working conditions. Many kinds of welded connections and some riveted connections are used that will give practically no local deformation and therefore come within the assumptions that have been made in the preceding chapters. The following analysis is based on the assumption of rigid connections, that is, that the end tangents of all members meeting at a joint rotate through the same angle. The problem of semirigid connections can be treated better after members with variable moments of inertia have been considered.

5.2 *Selection of the Primary Frames*

The frame shown in Fig. 5.1 is typical of those encountered in many buildings where torsional action is neglected and where the connections

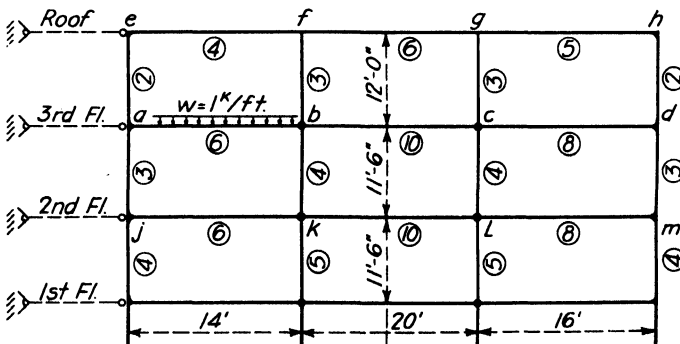



Fig. 5.1

undergo practically no local deformation. In such frames the vertical loads on the girders will be carried to the foundations largely by the adjacent columns. The end moments acting on any girder are affected by the amount of end restraint, which is primarily a function of the stiffness of the girder as compared to the stiffness of the adjacent members. Horizontal movement of the floors may be of importance in some structures, but for the present it will be assumed that the frame is supported against any sidesway, and, consequently, the fixed-end moments need be corrected only for the rotation of the joints. The removal of the restraining forces, if necessary, will be treated in the following chapter.

When a span of the frame in Fig. 5.1, such as *ab*, is loaded, the end moments in the various members, due to this load only, will be affected by the rotation of joints *a* and *b* and, to a much less degree, by the rotation of other joints such as *e*, *f*, *c*, *j*, and *k*. Frequently, the rotation of these other joints can be neglected or, in other words, the members can be considered as fixed at those points. A better approximation is often obtained by assuming that these joints do undergo some rotation, the amount of this rotation and its effect upon the moments of the various members being estimated by the conditions stated in the following derivation.

Let any member *ab* (Fig. 5.2) be subjected to a moment M_{ab} at the end *a*, and let the end *b* be restrained by the members *bc*, *bd*, *be* that are either fixed as at *c* and *d* or hinged as at *e*. Usually a fixed-end condition should be assumed.

Applying the slope deflection equations to the frame in Fig. 5.2 gives

$$M_{ab} = K_{ab}(4\theta_a + 2\theta_b) \qquad M_{ba} = K_{ab}(2\theta_a + 4\theta_b)$$

$$M_{bc} = 4K_{bc}\theta_b \qquad M_{bd} = 4K_{bd}\theta_b \qquad M_{be} = 3K_{be}\theta_b$$

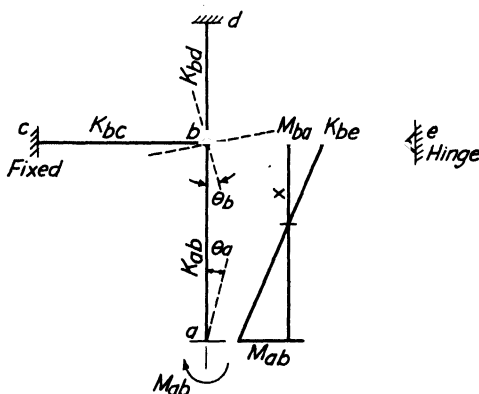



Fig. 5.2

Since $\Sigma M_b = 0$,

$$\theta_b(4K_{ab} + 4K_{bc} + 4K_{bd} + 3K_{be}) + 2K_{ab}\theta_a = 0$$

or, using the notation of Article 4.7,

$$\theta_b = -\frac{2K_{ab}}{\Sigma_b CK}\theta_a = -\tfrac{1}{2}r_{ba}\theta_a$$

where r_{ba} is the distribution factor for the member *ab* at the joint *b*. If this value of θ_b is now substituted in the preceding expressions for M_{ab} and M_{ba}, the following equations will be obtained:

$$M_{ab} = K_{ab}(4\theta_a - r_{ba}\theta_a) = (4 - r_{ba})K_{ab}\theta_a \qquad (5.1a)$$

$$M_{ba} = K_{ab}(2\theta_a - 2r_{ba}\theta_a) = 2(1 - r_{ba})K_{ab}\theta_a \qquad (5.1b)$$

Another convenient way of expressing M_{ba} is

$$M_{ba} = \frac{2(1 - r_{ba})}{4 - r_{ba}} M_{ab} \qquad (5.1c)$$

in which

$$\frac{2(1 - r_{ba})}{4 - r_{ba}}$$

is the carry-over factor from *a* to *b*. From equations 5.1*a* and 5.1*c*, the coefficient *C* at the end *a* and the carry-over factor from *a* to *b* can be obtained for any degree of restraint $(1 - r_{ba})$ at the end *b*. Thus, when the end *b* is considered fixed, r_{ba} equals 0, and when hinged, r_{ba} equals 1. The stiffness factor for the member *ab* at the end *a* can therefore vary from $3K$ to $4K$, and the carry-over factor from 0 to $\frac{1}{2}$ (see Fig. 5.3).

In the analysis of a building frame that is subjected to vertical loads, it is frequently convenient to analyze the structure for the loads on each span separately and then to combine the loadings so as to obtain the maximum and minimum moments. When only one span is loaded, such as *ab*, Fig. 5.1, it is often assumed that all joints except *a* and *b* remain fixed. In many problems, however, it is better to make use of equations 5.1*a* and 5.1*c*, which enable a more accurate analysis of the structure to be made. The part of the structure considered in the analysis will be designated the primary frame in which it may sometimes be necessary to include more of the structure than is used in this illustration. Equations 5.1*a* and 5.1*c* are not easily applied to members that have transverse loading.

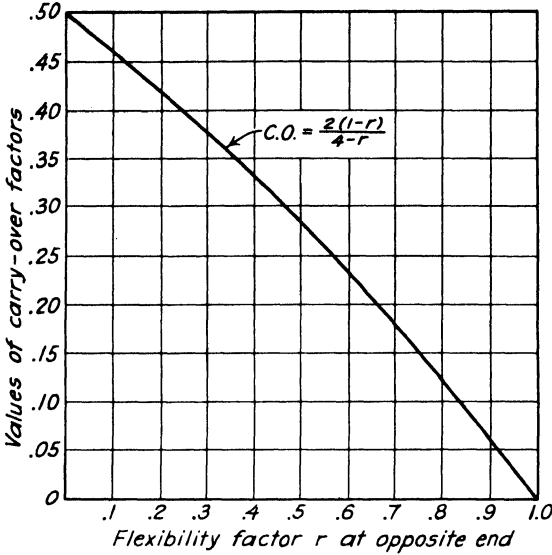



Fig. 5.3

5.3 *Analysis of the Primary Frame*

The end moments acting on the members of the frame in Fig. 5.4 will be calculated by the moment-distribution method, using the modified stiffness factor $(4 - r)K$ for all members, except ab. Although this procedure gives only a slight change in the distribution factor, it makes a considerable difference in the carry-over factors which affect the moments at the joints away from the loaded span. The distribution and

Fig. 5.4

carry-over factors for members at joints a and b when Equations 5.1a and 5.1c are used are

Joint a

Member	C	K	CK	r	Carry-over
ae	(4 − 0.333)	2	7.33	0.172	0.362
ab	4	6	24.0	0.563	0.5
aj	(4 − 0.23)	3	11.31	0.265	0.408
			$\Sigma CK =$ 42.64	1.000	

Joint b

Member	C	K	CK	r	Carry-over
ba	4	6	24.0	0.277	0.5
bf	(4 − 0.23)	3	11.31	0.131	0.408
bc	(4 − 0.4)	10	36.0	0.415	0.333
bk	(4 − 0.16)	4	15.36	0.177	0.437
			$\Sigma CK =$ 86.67	1.000	

The numerical solution for the end moments by the moment-distribution method is given in Fig. 5.5. The fixed-end moment at joint a, −16.33 ft-kips, is first distributed, and one-half the correction for M_{ab} is carried over to M_{Fba}; that is,

$$(\tfrac{1}{2})(9.20) = 4.6$$

is added to the fixed-end moment 16.33 at b.

The unbalanced moment of 20.93 at b is now distributed, and the correction −5.80 applied to M_{ba} produces a moment of −2.90 at a. Joint a is again balanced, and the procedure is continued until the desired accuracy is obtained.

Fig. 5.5

Fig. 5.6

After the moments at the joints a and b are determined, the other end moments can be obtained from the carry-over factors that have already been computed. Thus the moment at e will equal (0.362)(3.33), or 1.21, which is correct within the limits of the conditions represented by Fig. 5.4.

The moments caused by a uniform load of 1 kip per foot on the span bc are shown in Fig. 5.6. The solution of this problem is carried out like the preceding one; that is, the distribution factors for the members framing into joint b and c are calculated for a modified restraint for all members except bc. The carry-over factors have also been determined on the basis of a degree of flexibility at the far ends equal to r.

The moments at the ends of the members that provide the restraining action for the unloaded spans can also be obtained with sufficient accuracy from the preceding analysis. Thus, in Fig. 5.5, the moment M_{cb} (-3.0) is resisted by a $+3.0$ in the members cg, cd, and cl (Fig. 5.4) which is distributed to the three members in proportion to their K values. The value of M_{cd} will therefore be (8/15)(3.0), or 1.6, and M_{dc} will be (0.225)(1.6) or 0.36. This use of modified restraints in determining the distribution of moments can be extended further, but, in general, for any further extension the law of diminishing returns takes over.

This arrangement provides a convenient solution for the end moments acting on the members of a continuous frame due to any type of vertical loads applied to a particular girder. A numerical solution for each primary frame can be conveniently recorded on standard size paper.

5.4 *Maximum and Minimum Moments*

The preceding analysis indicates that, although a vertical load on one span of a building frame will produce moments over several bays and in the adjacent stories, in most practical problems only the moments in the adjacent beams and columns need be considered. From the moments that have been computed for a unit load on each span, the combined effect of the dead load of the structure and of the live load (for full span loadings) can be obtained by superposition. However, although only the end moments have been computed thus far, it is necessary that the designer also investigate the moments that may occur at any intermediate section. Such moments can be pictured graphically both by drawing moment diagrams for loads on various spans and by constructing influence diagrams for the moments at various sections of the girders.

In Fig. 5.7*a*, the moment diagram is shown for a load P on the span *ab*, and in Fig. 5.7*b* for a load on span *bc*. A study of these moment diagrams yields some interesting information; for instance, when the load P is applied anywhere on span *ab*, the moment diagrams for spans *bc* and *cd* pass through the points f_{2r} and f_{3r}, respectively. These two points, which

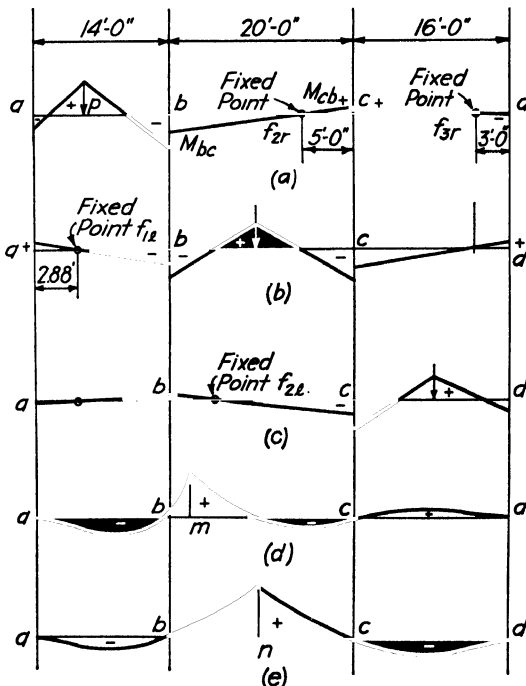



Fig. 5.7

are commonly called "fixed points," are sometimes used in the analysis of continuous frames. The distance x of the fixed points from the adjacent support can be obtained directly from equation 5.1*c*, thus

$$\frac{M_{cb}}{M_{bc} + M_{cb}} = \frac{x}{L}$$

and

$$M_{cb} = \frac{2(1 - r)}{4 - r} M_{bc}$$

From these expressions the distance x of a fixed point from an end whose degree of flexibility is r is given by the expression

$$x = \frac{2(1 - r)}{3(2 - r)} L \tag{5.2}$$

For all practical purposes, the value of r can be taken equal to the ordinary distribution factor.

Each span will have two fixed points that are useful in determining the nature of the moments produced in that span by loads on adjacent spans. For convenience, any span such as *bc* can be considered in three parts: the portion between the fixed points f_{2l} and f_{2r}, which is subjected only to negative moments from either adjacent span; the portion between the fixed point f_{2r} and the support *c*, which is subjected to positive moments from loads on span *ab* and negative moments from loads on span *cd*; and the portion between f_{2l} and the support *b*, which is subjected to negative moments from loads on span *ab* and positive moments from loads on span *cd*.

The most important features of the diagrams in Fig. 5.7 can be summarized as follows.

1. Influence diagrams for moment for sections between a fixed point and the support are of the type shown in Fig. 5.7*d* for point *m*.
2. Influence diagrams for sections between the fixed points are of the type shown in Fig. 5.7*e* for point *n*.
3. For points such as *m*, the maximum and minimum moments are obtained for partial loading of the span itself and with one adjacent span loaded.
4. For point *n*, the maximum moments occur when the span itself is fully loaded together with alternate spans. The moment caused by the loads on the alternate spans can usually be neglected as it is comparatively small. The minimum moment at point *n* occurs with no live load on the span *bc* but with the adjacent spans *ab* and *cd* fully loaded.

5. If each span is loaded separately with a unit uniform load as in the preceding analysis, the separate values for dead and live loads can be combined to give actual maximum and minimum values for all sections between the fixed points and at the supports, but only approximate values for sections between the supports and the fixed points.

Example 5.1 The maximum and minimum moment curves will be drawn for the spans *ab* and *bc* of the frame in Fig. 5.1 for a uniform dead load of 2 kips per foot and a uniform live load of 4 kips per foot. The end moments for a unit uniform load on spans *ab* and *bc* have already been calculated in Figs. 5.5 and 5.6. Therefore the moments at intermediate points can

Table 5.1

Point 1	y_1 2	y_2 3	y_3 4	$2\Sigma y$ 5	$4\Sigma(+y)$ 6	$4\Sigma(-y)$ 7	Max 8	Min 9
6*r*	−9.02	−23.06	3.52	−57.12	14.08	−128.32	−43.04	−185.44
7	−7.0	4.50	1.08	−2.84	22.32	−28.00	19.48	−30.84
f2l	−6.12	13.00	0	13.76	52.0	−24.48	65.8	−10.72
8	−5.0	20.91	−1.38	29.06	83.64	−25.52	112.70	3.54
9	−3.0	26.12	−3.83	38.58	104.48	−27.32	143.06	11.26
10	−1.0	20.33	−6.29	26.08	81.32	−29.08	107.40	−3.00
f2r	0	13.0	−7.50	11.0	52.0	−30.0	63.0	−19.0
11	1.0	3.34	−8.74	−8.80	17.36	−34.96	8.56	−43.76
12	3.0	−24.75	−11.20	−65.90	12.00	−143.80	−53.90	−209.70

be computed from the conditions for statical equilibrium or by scaling the ordinates to the bending-moment diagrams. The effects of loads on the floors above and below will be neglected, although they can readily be included, and for the exterior spans it is sometimes necessary to do so. In Table 5.1 the ordinates y_1, y_2, y_3 give the values of the moments at various sections in span *bc* for a uniform load of 1 kip per foot on spans *ab*, *bc*, and *cd*, respectively. The dead load moment is given by column 5, and the maximum and minimum live load moments by columns 6 and 7. The maximum and minimum combined values are recorded in columns 8 and 9.

These values are shown graphically by the curves in Fig. 5.8. The solid portions of the curves between the fixed points represent actual maximum and minimum values; the dotted portions are approximate. The error involved in the dotted portion, however, is relatively small.

From studying the curves for maximum and minimum moments (Fig. 5.8), it can be seen that large variations in bending moments are obtained at the various sections and that the position of the points of contraflexure varies over a considerable distance. The amount of this

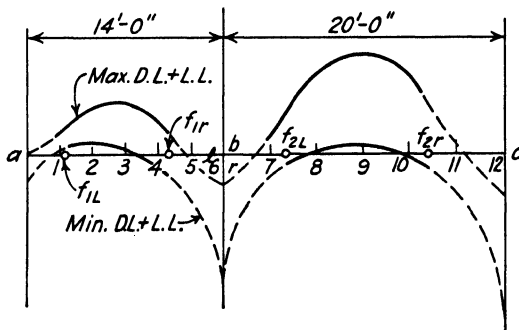



Fig. 5.8

variation depends on the arrangement of the spans and on the ratio of the live load to the dead load.

5.5 *Maximum Shear*

The maximum shear that can occur at any section of a girder in a continuous building frame is largely due to the loads on that span except for unusual span arrangements. Nevertheless the effect of continuity on the shear should be studied, at least to the extent of knowing the error involved if the loads on adjacent spans are neglected. The influence diagram for the shear at any section *m* of an interior girder in a continuous frame, as shown in Fig. 5.9, is useful in studying the problem. The shape of the influence diagram can be easily sketched by using the reciprocal theorem as illustrated in Chapter 2. The problem of determining the maximum shear is studied best by considering first, the shear due to loads on the span itself, and second, the shear from loads on one of the adjacent spans.

The influence diagram in Fig. 5.9 shows that the maximum positive live load shear for a uniform load on span *bc* occurs when the load is placed on part of the beam. This condition is the same as for a simply supported beam. The effect of end restraint on the maximum shear is

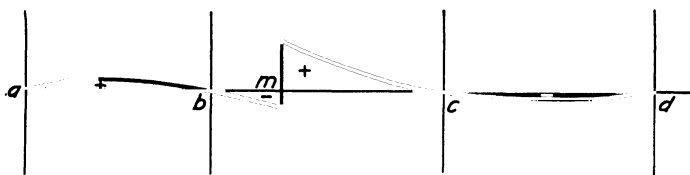



Fig. 5.9

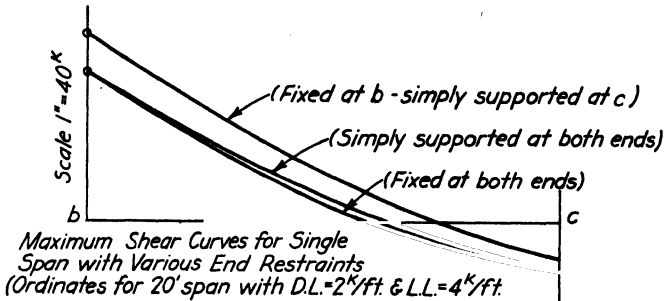



Fig. 5.10

shown in Fig. 5.10, in which maximum shear curves have been drawn for uniform loading on the span *bc* for three different end conditions, namely, simply supported at both ends, fixed at both ends, and simply supported at one end and fixed at the other. These curves were drawn for a live load equal to twice the dead load. A study of these maximum shear curves shows that the difference in end restraints and not the degree of restraint is the more important factor. The values for both ends simply supported differ but slightly from those for both ends fixed, whereas when one end is simply supported and the other fixed, there is a noticeable difference. This comparison indicates that, for interior spans, the shear from loads on the span itself is practically the same as for a simply supported beam, whereas for exterior spans some allowance must be made for the increased shear at the interior support. An increase of 20% over the maximum shear for a simply supported span is usually a sufficient allowance for the end shear, but at intermediate sections a larger allowance should be made.

The shear due to loads on adjacent spans can be calculated directly from the end moments that have been determined. This effect is relatively small unless the adjacent spans are large as compared to the span itself. A more important action is the fluctuation in the shear values that takes place if any unequal settlement of the supports occurs, and consequently some reserve strength to allow for such redistribution of shear is desirable.

5.6 *Maximum Stress in Columns*

Since the end shear in the girders must be resisted by the axial stress in the columns, the latter can be determined directly from the shear values. The maximum axial load in any column will be obtained when the adjacent bays are fully loaded in all stories. For exterior columns such loading will also give large bending moments, whereas for interior columns the maximum bending moments will be obtained with a different

loading arrangement. In other words, maximum direct stress and maximum bending moment do not occur simultaneously for interior columns. The most critical stress conditions will vary for different span arrangements.

The maximum moment in an exterior column such as *aj* (Fig. 5.1) can be calculated with sufficient accuracy by loading spans *ab* and *jk*; for the interior columns, such as *bk*, by loading *bc* and *kl*. The numerical values can be obtained by combining the moments that have already been computed for the loads on each span.

Example 5.2 The maximum value of V_{ab}, V_{ba}, and V_{bc} for the building frame in Fig. 5.1 will be computed for a dead load of 2 kips per foot and a live load of 4 kips per foot, and approximate maximum shear curves will be drawn.

The maximum value of V_{ab} occurs with both dead load and live load on spans *ab* and *cd*, but with only dead load on *bc*. By using the end moments that are recorded in Figs. 5.5 and 5.6 for spans *ab* and *bc* and by making similar calculations for a uniform load on *cd*, the following values are obtained:

From span *ab*

$$V_{ab} = 42.0 - \frac{(6)(15.72 - 8.45)}{14} = 38.9$$

From span *bc*

$$V_{ab} = -\frac{(2)(10.1 + 2.64)}{14} = -1.8$$

From span *cd*

$$V_{ab} = \frac{(6)(1.63 + 0.42)}{14} = 0.9$$

$$\text{Total } V_{ab} = 38.0$$

For maximum negative value of V_{ba}, full dead loads and live loads are placed on spans *ab* and *bc*, whereas only the dead load is placed on span *cd*.

From span *ab*

$$V_{ba} = -42 - \frac{(6)(15.72 - 8.45)}{14} = -45.1$$

From span *bc*

$$V_{ba} = -\frac{(6)(10.1 + 2.64)}{14} = -5.5$$

From span *cd*

$$V_{ba} = +\frac{(2)(1.63 + 0.42)}{14} = 0.3$$

$$\text{Total } V_{ba} = -50.9$$

For maximum value of V_{bc}, the dead load and live load are placed on spans *ab* and *bc* only, whereas the dead load is on *cd*.

From span *ab*

$$V_{bc} = \frac{(6)(9.02 + 3.0)}{20} = 3.6$$

From span *bc*

$$V_{bc} = 60 - \frac{(6)(24.7 - 23.1)}{20} = 59.5$$

From span *cd*

$$V_{bc} = -\frac{(2)(11.2 + 3.52)}{20} = -1.5$$

$$\text{Total } V_{bc} = 61.6$$

These values of V_{ab}, V_{ba}, and V_{bc} vary by -9.5%, $+19.8\%$, and $+2.7\%$, respectively, from the corresponding values that would be obtained for simply supported beams.

The maximum positive and negative shear at the center of span *ab* for the live load on span *ab* can be determined with sufficient accuracy by taking the maximum positive live load shear as one-fourth of the live load end shear V_{ab}, and the negative shear as one-fourth of the live load end shear V_{ba}. From the preceding calculations, the live load end shears for a uniform load of 4 kips per foot are the following.

At end *a*

$$\tfrac{4}{6}(38.9) = 26.0 \qquad \text{therefore} \quad +V_{\mathbb{C}} = \frac{26.0}{4} = 6.5$$

At end *b*

$$\tfrac{4}{6}(-45.1) = -30.0 \qquad \text{therefore} \quad -V_{\mathbb{C}} = \frac{-30.0}{4} = -7.5$$

Since the dead load end shear V_{ab} is 13.0 kips, the dead load shear at the center is

$$13.0 - (2)(7.0) = -1.0 \text{ kip}$$

Since the shear at the center of span *ab* for loads on the other spans *bc* and *cd* is the same as for the end shears V_{ab} and V_{ba}, the combined values give

$$V_{\mathbb{C}} = 6.5 - 1.0 - 1.8 + 0.9 = 4.6 \qquad \text{(3.2 by exact analysis)}$$

$$V_{\mathbb{C}} = -7.5 - 1.0 - 5.5 + 0.3 = -13.7 \qquad \text{(}-13.5\text{ by exact analysis)}$$

The typical maximum shear curves illustrated in Fig. 5.10 show that a straight line is a close approximation to the actual curve, and therefore a linear variation is used for this problem (see Fig. 5.11).

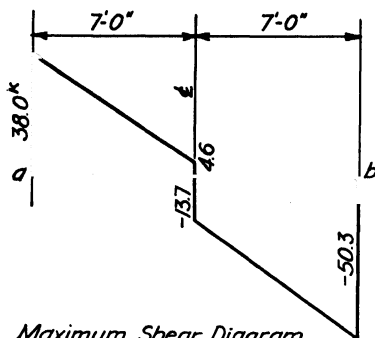



Fig. 5.11

5.7 Selection of K Values

The foregoing procedure for calculating the maximum moments and shears in continuous building frames requires the use of the K or I/L value of each member. Therefore estimated proportions of the various members must be made before the final design can be started. If the preliminary design is carefully performed, no further changes in the K values should be necessary. As in most design work, experience is an important factor in selecting trial proportions and arrangements. However, with the aid of some approximate calculations and with some knowledge of the variation of moments and shears throughout a building frame, even a relatively inexperienced designer should be able to choose sufficiently correct proportions. It should be remembered that small variations in the K values are not important in the determination of moments and shears.

In most building frames of slab, beam, and column construction, a selection of the preliminary proportions for the beams and columns can be facilitated by the following trial calculations.

1. Calculate the maximum end shears for the simply supported beams and girders. For exterior spans increase the interior end shear by 15 to 20%. The end shears in interior spans will usually be increased by less than 5%, which can be used as a trial value.
2. Calculate the fixed-end moments for the loading conditions giving maximum negative bending moments at the supports. Using estimated proportions that will satisfy the shear requirements, headroom restrictions, and deflection make one distribution cycle for the exterior joints followed by one distribution at the interior joints. The sizes required for negative moments can be estimated from these values.

3. Give some consideration to the size and placing of the reinforcing steel in reinforced concrete beams. In proportioning tee-beams the maximum positive moments near the center of the span are not usually critical except in the design of joists.
4. After the girders on one floor have been designed, they will provide useful information for the selection of preliminary proportions for the beams and girders on other floors. Always keep in mind possible duplication of forms.
5. A trial value for column sizes can often be made from an estimate of the total axial load on the column. Frequently, this information has already been tabulated for a preliminary estimate of footing sizes.
6. The preliminary proportioning of members as well as the final design requires a basic understanding of the properties of materials and the requirements of the various building codes and specifications. In addition, the functional use of the building must always be considered.

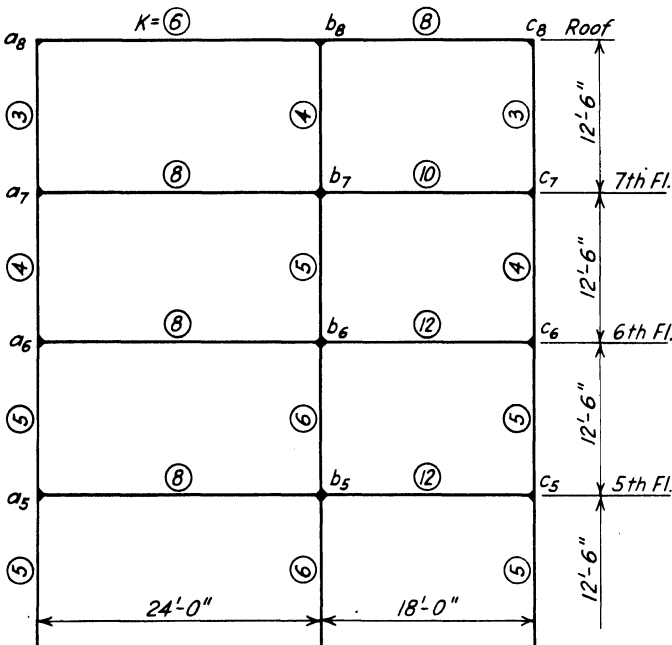
5.8 Summary

The preceding discussion of the analysis of building frames under the action of vertical loads is concerned with a general method of solution rather than a detailed procedure for design calculations. The application of this method of analysis to any building frame will naturally require a careful study of some of the following factors.

1. The distribution of the load in any bay to the various two-dimensional frames into which the building is divided. This distribution depends on the bending and torsional resistance of the floor and its relation to the members of the various frames. The recommendations of the various building codes should be studied with respect to this problem.
2. A selection of relative I/L values based upon an estimate of the critical conditions for various members. In the preliminary design, it should be noted that the shear in the beams and the direct stress in the columns will not be affected greatly by the end moments. If haunched members are used, more difficulty may be encountered in making a preliminary design, particularly for built-up steel sections. For reinforced concrete structures the recommendations of the A.C.I. Building Code on assumptions for design should be studied.
3. Any unusual conditions should be noted and provided for in the preliminary design through an approximate analysis or from experience.

4. The designer should always remember that the moments that are computed are but one factor in the design, and that only reasonable accuracy in their determination is essential. A considerable variation in the stiffness of a member is required to produce an appreciable change in the moments. The designer must be able to recognize in advance the relative importance of various factors.

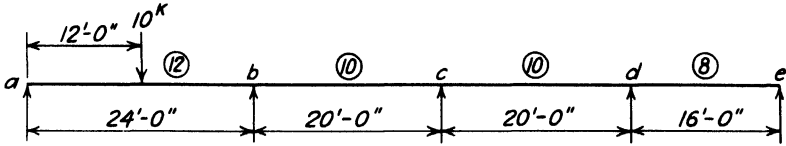
Problems

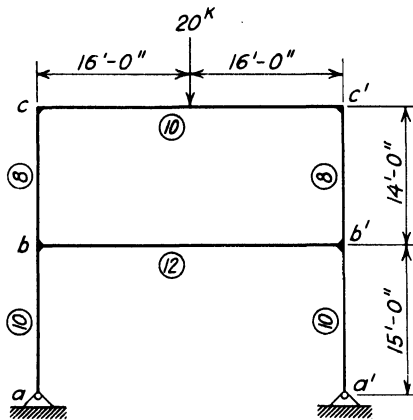
5.1 Calculate the end moments in beams a_7b_7 and b_7c_7 and in all columns above and below the seventh floor for a uniform load of 1 kip per linear foot on span a_7b_7. Do the same for a unit uniform load on span b_7c_7. Construct the bending-moment diagrams for each loading and determine the position of the fixed-points in the beams. Use the distribution and carry-over factors based upon the modified end restraint and the K values given in the circles.

Problem 5.1

5.2 From the results of Problem 5.1, construct maximum and minimum moment curves for spans a_7b_7 and b_7c_7 for a dead load of 1.6 kips per foot and a live load of 2.5 kips per foot.

5.3 For the same dead and live load as in Problem 5.2, construct the maximum and minimum moment curves for column a_8a_7 of the frame in Problem 5.1. Consider the effects of loads on the sixth and seventh floors only.

5.4 Construct a maximum shear diagram for span a_7b_7 by the procedure used in Example 5.2. Use the same dead and live loads as in Problem 5.2.

5.5 Calculate all end moments for the continuous beam shown by the method of modified end restraints. *K* values are recorded on the sketch.

Problem 5.5

5.6 Calculate the end moments for all members by the moment-distribution method using modified end restraints that take advantage of symmetry whenever possible. *K* values are recorded on the diagram.

Problem 5.6

5.7 Solve Problem 5.6 if the single 20 kips load is applied at the center of member *bb'*.

References

1 L. T. Evans, "Modified Slope-Deflection Equations," *J. Am. Conc. Inst.*, October, 1931.

2 T. F. Hickerson, *Structural Frameworks*, University of North Carolina Press, 1934.

3 Ernst Suter, *Methode der Festpunkte*, Springer-Verlag, Berlin.

4 T. Y. Lin, "A Direct Method of Moment Distribution," *Trans. Am. Soc. C. E.*, Vol. 102 (1937).

5 J. A. Wise, "Precise Moment Distribution Method," *J. Am. Conc. Inst.*, May 1939.

6 A. Amerikian, *Analysis of Rigid Frames (An Application of Slope-Deflection)*, United States Government Printing Office, 1942.

7 E. B. Russell, *Analysis of Continuous Frames by the Method of Restraining Stiffnesses*, Ellison and Russell, San Francisco, California, 1934.
8 Ralph W. Stewart, "Relative Flexure Factors for Analyzing Continuous Structures," *Trans. Am. Soc. C. E.*, Vol. 104 (1939).
9 L. H. Nishkian and D. B. Steinman, "Moments in Restrained and Continuous Beams by the Method of Conjugate Points," *Trans. Am. Soc. C. E.*, Vol. 90 (1927).
10 *Continuity in Concrete Building Frames*, Portland Cement Assoc., Chicago, Illinois.
11 W. S. Londe and M. F. James, *Concrete Engineering Handbook*, Section 9 McGraw-Hill Book Co.

6

Continuous Frames with Joints Having Different Linear Displacements

THE USE OF AUXILIARY FORCE SYSTEMS TO CONTROL TRANSLATION OF THE JOINTS

6.1 Nature of Auxiliary Forces

When the joints of continuous-frame structures undergo motion of translation as well as rotation, direct methods of successive approximation are not easily applied, for the increments may then converge slowly if at all. Failure to obtain convergence is largely due to the fact that the translation of any joint will have an important effect on the end moments of many members, whereas the rotation of a joint affects primarily the contiguous members. For this reason the analysis of many continuous frames by methods of successive approximation, such as the moment-distribution method, is practical only if the translation of the joints is controlled by the use of auxiliary force systems. The nature of such auxiliary force systems will necessarily depend on the motion that they must give or prevent in the structure, but, for any problem, the summation of the auxiliary force systems and their displacements must be made equivalent to the actual force system and the actual movement of the frame.

The use of an auxiliary force system in controlling one displacement has already been illustrated by the analysis of the frame in Fig. 4.17. The application of this method of procedure to frames with several

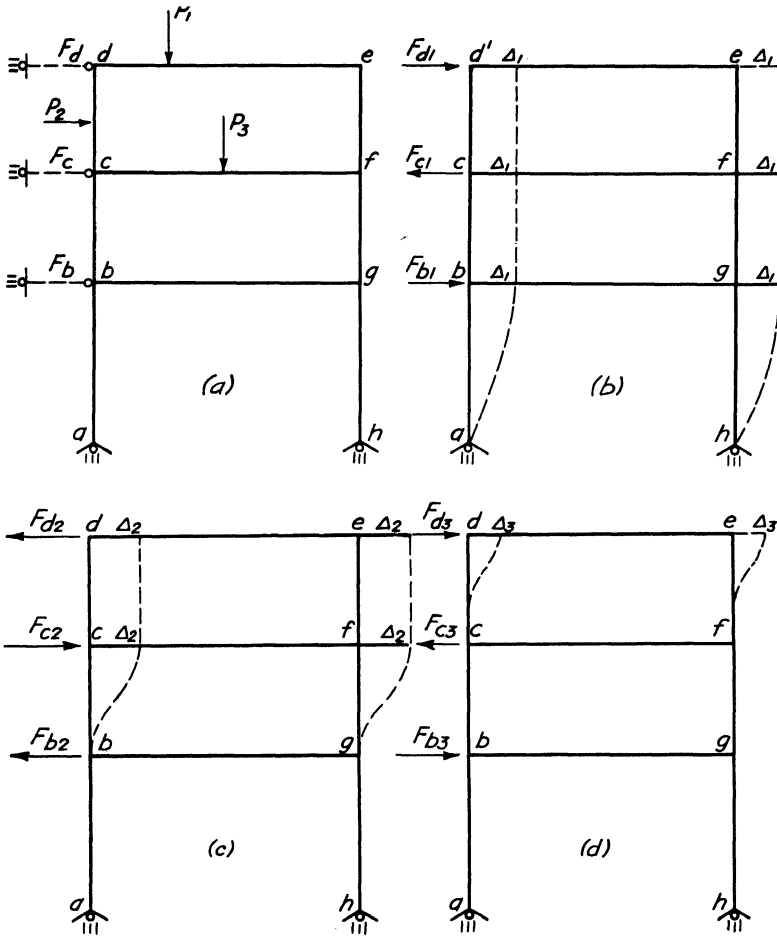



Fig. 6.1

displacements will now be discussed. A common example is the frame in Fig. 6.1*a*, which is acted upon by forces P_1, P_2, P_3. To prevent any horizontal displacements of the joints it will be necessary to apply a system of auxiliary forces F_b, F_c, F_d, as shown by the dotted lines. The change in length of the members due to axial stress is, of course, neglected. The analysis by the moment-distribution method can now proceed in the ordinary manner, and the values of F_b, F_c, and F_d can be calculated. To remove these forces, three different auxiliary force systems, as shown by Figs. 6.1*b*, *c*, and *d*, will be required, each of which permits but one horizontal displacement. The algebraic relation existing between any single elastic displacement Δ and the force system producing it will then

be known if it is determined for any assumed value of Δ, for the relationship is a linear one. For this reason, each auxiliary force system and the shears and moments which it produces can be expressed in terms of one displacement Δ, and the summation of all auxiliary force systems can be expressed as functions of the various Δ values. The numerical values of the displacements Δ are calculated from the necessary equilibrium conditions. As an example of such equilibrium conditions the forces in Figs. 6.1*b*, *c*, and *d*, when combined, must remove the virtual loads F_b, F_c, and F_d used in Fig. 6.1*a*, or, expressed algebraically (with appropriate signs),

$$\begin{aligned} F_{b1} + F_{b2} + F_{b3} + F_b &= 0 \\ F_{c1} + F_{c2} + F_{c3} + F_c &= 0 \\ F_{d1} + F_{d2} + F_{d3} + F_d &= 0 \end{aligned} \tag{6.1}$$

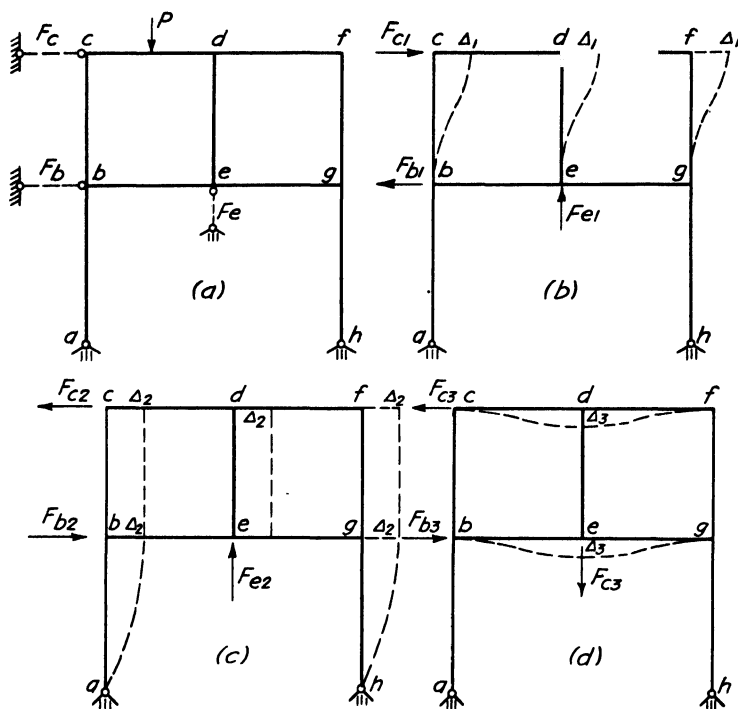
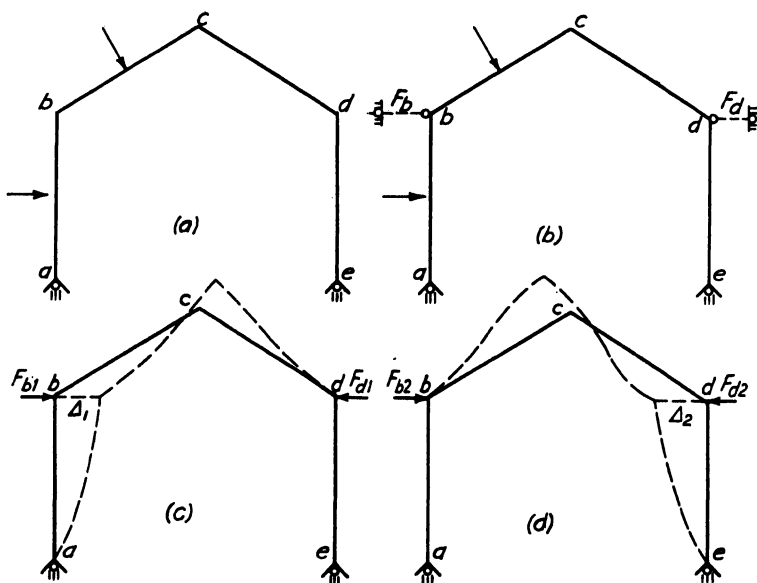
Although these equations are sufficient to determine the true value of Δ_1, Δ_2, and Δ_3, it will be shown later that the numerical calculations are often simplified if the equilibrium conditions are taken with respect to sections through the structure rather than around the joints.

Further examples of auxiliary force systems are given by the frame in Fig. 6.2*a*, which is subjected to the load P and will require the auxiliary forces F_b, F_c, and F_e to prevent translation of the joints. These forces are removed by combining the auxiliary force systems shown by Figs. 6.2*b*, *c*, and *d*, each of which is expressed by a single value of Δ. The values of Δ_1, Δ_2, and Δ_3 are obtained from the equilibrium conditions that will be explained in a numerical example.

In a similar manner, the frame of Fig. 6.3*a* can be solved by combining the solutions of the force systems shown in Figs. 6.3*b*, *c*, and *d*. (See Article 3.8 for the motion of joint *c*.)

Example 6.1 The foregoing procedure for controlling the translation of the joints will be followed in solving for the moments in the frame of Fig. 6.4. The solution can be made by the moment-distribution method if the horizontal movement of the structure is considered in terms of three separate quantities, Δ_1, Δ_2, and Δ_3, as illustrated in Figs. 6.1*b*, *c*, and *d*. Let us first assume that $E\Delta_1$ is equal to 10 kips per foot and all rotations are zero except at the supports *a* and *a'*. Then, the only members subjected to flexure are *ab* and *a'b'*, which will be acted on by fixed-end moments at *b* and *b'* equal to (see equation 4.11, Article 4.5)

$$M_{Fba} = M_{Fb'a'} = -\frac{3EI}{L}\frac{\Delta_1}{L} = -\frac{(3)(12)(10)}{16} = -22.5 \text{ ft-kips}$$

(The value of K is also taken in foot units, although this assumption is not essential if the actual motion is not desired.)

The fixed-end moments due to any translation Δ can also be represented by the term M_F, for these moments are always kept separate from those due to transverse loads on the members. These fixed-end moments due to an assumed value of Δ_1 are corrected for rotation of the joints by the moment-distribution method as indicated by the numerical operation in Fig. 6.5. In this problem considerable numerical work is saved by taking advantage of the symmetry of the structure. The slope deflection equation for the moment $M_{bb'}$, which is

$$M_{bb'} = (4)(14)\theta_b + (2)(14)\theta_b$$

can be written

$$M_{bb'} = (6)(14)\theta_b = 84\theta_b$$

Fig. 6.4

since by symmetry we know that $\theta_{b'}$ is equal to θ_b. Therefore when joint b is permitted to rotate an amount θ_b that is necessary to balance the moments around the joint, joint b' is assumed to rotate the same amount, that is, joints b and b' are rotated at the same time. Joints a and a' are free to rotate, whereas the remaining joints are kept in a fixed

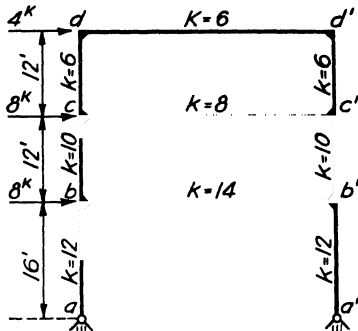
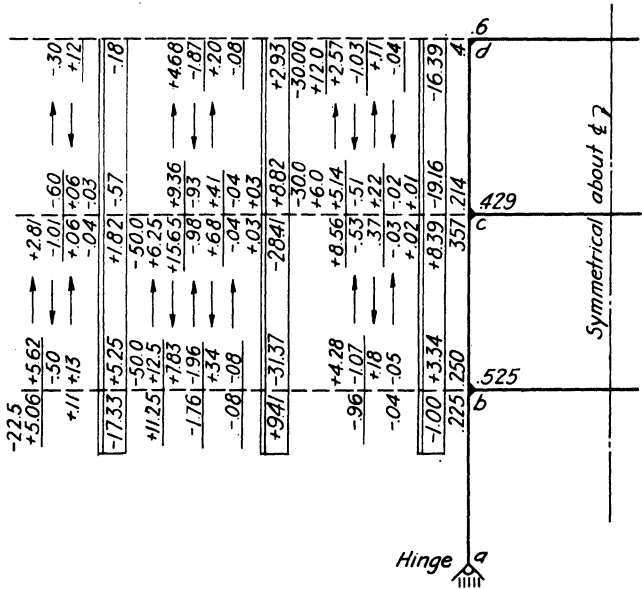



Fig. 6.5

condition. The moments around the joint b due to any rotation θ_b and $\theta_{b'}$ will therefore be

$$M_{ba} = (3)(12)\theta_b = 36\theta_b \qquad M_{bb'} = 84\theta_b \qquad M_{bc} = (4)(10)\theta_b = 40\theta_b$$

The distribution factors at joint b will be

$$r_{ba} = \frac{36}{160} = 0.225 \qquad r_{bb'} = \frac{84}{160} = 0.525 \qquad r_{bc} = \frac{40}{160} = 0.25$$

In the same manner, the distribution factors at joint c can be computed.

$$r_{cb} = \frac{40}{112} = 0.357 \qquad r_{cc'} = \frac{48}{112} = 0.429 \qquad r_{cd} = \frac{24}{112} = 0.214$$

and at joint d

$$r_{dc} = \frac{24}{60} = 0.4 \qquad r_{dd'} = \frac{36}{60} = 0.6$$

The distribution of the fixed-end moments is carried out in the usual manner as indicated by the arrows in Fig. 6.5. After the end moments have been determined, the value of the shear in each panel and the value of the auxiliary forces can be calculated. Thus the value of each panel shear, due to Δ_1, is

PANEL 1

$$V_1' = \frac{(2)(17.33)}{16} = 2.17 \qquad \text{or} \qquad 0.217\Delta_1 \rightarrow$$

PANEL 2

$$V_2' = -\frac{(2)(5.25 + 1.82)}{12} = -1.18 \qquad \text{or} \qquad -0.118\Delta_1 \leftarrow$$

PANEL 3

$$V_3' = \frac{(2)(0.75)}{12} = 0.125 \qquad \text{or} \qquad 0.0125\Delta_1 \rightarrow$$

In a similar manner, fixed-end moments of -50 ft-kips in panel 2, due to an assumed movement of $E\Delta_2$ equal to 10, are distributed and the shear in each panel calculated. The shear values will be

$$V_1'' = -0.118\Delta_2 \qquad V_2'' = 0.995\Delta_2 \qquad V_3'' = -0.196\Delta_2$$

For a horizontal movement $E\Delta_3$ equal to 10, fixed-end moments of $-(6)(6)(10)/12 = -30$ ft-kips will occur in panel 3. After these moments are distributed, the following shears in each panel are obtained:

$$V_1''' = 0.0125\Delta_3 \qquad V_2''' = -0.196\Delta_3 \qquad V_3''' = 0.592\Delta_3$$

The correctness of these results is practically assured by the fact that they satisfy the reciprocal theorem. If the reciprocal theorem is applied to the forces in Figs. 6.1*b* and *d*, the following equation is obtained (see Article 2.5):

$$F_{d1}\Delta_3 = (F_{d3} - F_{c3} + F_{b3})\Delta_1$$

or

$$V_3'\,\Delta_3 = V_1''''\cdot\Delta_1$$

Using these results,

$$0.0125\Delta_1\cdot\Delta_3 = 0.0125\Delta_3\cdot\Delta_1$$

In the same way for the forces in Figs. 6.1*c* and *d*,

$$-F_{d2}\Delta_3 = (F_{d3} - F_{c3})\Delta_2$$

or

$$V_3''\Delta_3 = V_2''''\Delta_2$$

which gives for the preceding numerical values

$$-0.196\Delta_2\Delta_3 = -0.196\Delta_3\Delta_2$$

If the shears in each panel due to the auxiliary force systems are now set equal to the actual shear of the applied forces, a set of linear equations in terms of Δ_1, Δ_2, and Δ_3 is obtained.

PANEL 1

$$+0.217\ \Delta_1 - 0.118\Delta_2 + 0.0125\Delta_3 = 20$$

PANEL 2

$$-0.118\ \Delta_1 + 0.995\Delta_2 - 0.196\ \Delta_3 = 12$$

PANEL 3

$$+0.0125\Delta_1 - 0.196\Delta_2 + 0.592\ \Delta_3 = 4$$

Since in each equation the coefficient on the main diagonal is greater than the sum of the absolute values of the other coefficients in the equation, a solution for the Δ values by a method of iteration is practical. One method for solving such simultaneous equations is of sufficient importance to warrant a detailed description of the procedure.

The equations will be rearranged by solving the first for Δ_1, the second for Δ_2, and the third for Δ_3, giving

$$\Delta_1 = 92.2 + 0.543\Delta_2 - 0.058\Delta_3$$

$$\Delta_2 = 12.1 + 0.119\Delta_1 + 0.197\Delta_3$$

$$\Delta_3 = 6.75 - 0.021\Delta_1 + 0.331\Delta_2$$

If we first assume that Δ_2 and Δ_3 are zero, a value of 92.2 is obtained as a first trial value for Δ_1. This value of Δ_1 is then put in the next equation

for Δ_2, which gives

$$\Delta_2 = 12.1 + (0.119)(92.2) + 0 = 23.0$$

These values of Δ_1 and Δ_2 are then inserted in the equation for Δ_3, giving

$$\Delta_3 = 6.75 - (0.021)(92.2) + (0.331)(23.0) = 12.4$$

A new value of Δ_1 can now be obtained by using these values of Δ_2 and Δ_3.

$$\Delta_1 = 92.2 + (0.543)(23.0) - (0.058)(12.4) = 104.0$$

This procedure is continued until the desired accuracy is obtained. The values are tabulated in Table 6.1, from which it can be seen that the

Table 6.1

Cycle	$E\Delta_1$	$E\Delta_2$	$E\Delta_3$
1	92.2	23.0	12.4
2	104.0	26.8	13.44
3	106.0	27.2	13.52
4	106.2	27.3	13.55

convergence is rapid. In this case three cycles are sufficient for all practical purposes.

Because the moments in Fig. 6.5 are for $E\Delta$ values of 10, the moments (in foot-kips) for the above values of $E\Delta$ can be obtained by proportion, thus

$$M_{ba} = (-1.733)(106.2) + (0.941)(27.3) - (0.100)(13.55) = -159.8$$

$$M_{bc} = (0.525)(106.2) - (3.137)(27.3) + (0.334)(13.55) = -25.4$$

$$M_{cb} = -46.8 \qquad M_{cd} = -8.0 \qquad M_{dc} = -16.1$$

Example 6.2 The frame shown in Fig. 6.6*a* will be analyzed by the procedure already discussed for Fig. 6.2. Here the vertical load of 21 kips applied on the member *cd* will produce a rotation and displacement at each joint. If the change in length of the members is neglected, three different linear displacements, two horizontal and one vertical, must be considered. These joint displacements can be prevented by applying auxiliary forces at *b*, *c*, and *e* as illustrated in Fig. 6.6*a*. When the frame is held in this position, the fixed-end moments

$$M_{Fcd} = -56.0 \text{ ft-kips}$$

and

$$M_{Fdc} = +28.0 \text{ ft-kips}$$

produced by the concentrated load of 21 kips are distributed in the usual manner. The distribution procedure as well as the final moments are given on the diagram. After the end moments have been determined, the end shears in each member are calculated and from these values the magnitude of each auxiliary force is obtained. For instance, the force cc' must be equal to the algebraic sum of the shears in members bc, de, and fg, or

$$cc' = \frac{30.9 + 10.0 - 9.9 - 3.1 + 4.4 + 1.0}{12} = 2.78 \text{ kips} \leftarrow$$

Similarly,

$$bb' = \frac{-6.0 - 1.4}{16} - 2.78 = -3.24 \text{ kips} \rightarrow$$

$$ee' = V_{dc} + V_{df} + V_{eb} + V_{eg} = +6.74 + 1.16 - 0.19 - 0.17 = 7.54 \text{ kips} \uparrow$$

To remove these auxiliary forces, the procedure already explained for Figs. 6.2*b*, *c*, and *d* is used, that is, giving the structure three separate displacements Δ_1, Δ_2, and Δ_3. Let us first assume $\Delta_1 = 12$ units to the right; then ($E = 1$)

$$M_{Fbc} = M_{Fcb} = -\frac{(6)(10)(12)}{12} = -60$$

$$M_{Fed} = M_{Fde} = -\frac{(6)(6)(12)}{12} = -36$$

In the distribution of these moments (Fig. 6.6*b*), advantage can be taken of the symmetry of the structure to reduce the numerical work.

For Δ_1 equal to 12 units, the auxiliary forces are

$$cc' = \frac{(-26.7 - 31.2)(2) - 31.2 - 29.4}{12} = -14.70 \quad \text{or} \quad 1.225\Delta_1 \rightarrow$$

$$bb' = \frac{14.4 + 14.4}{16} + 14.70 = 16.50 \quad \text{or} \quad 1.375\Delta_1 \leftarrow$$

$$ee' = 0$$

If a value of Δ_2 equal to 16 units is assumed (see Fig. 6.6*c*), and if the joints are held without rotation, there will be fixed-end moments in members *ab* and *gh* equal to

$$M_{Fba} = M_{Fgh} = -\frac{(3)(12)(16)}{16} = -36$$

The joints are now permitted to rotate in the following order: *b, c, g, f, e, d.* It will be found that, owing to symmetry, only the joints *b*, *c*, *d*, and *e* need be used in the distribution of the moments. The numerical operations and the final moments are given on the diagram.

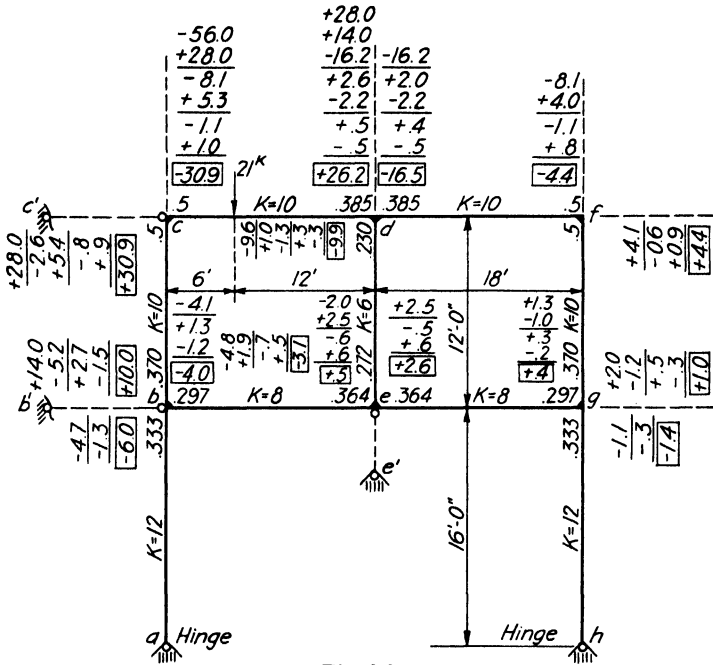



Fig. 6.6a

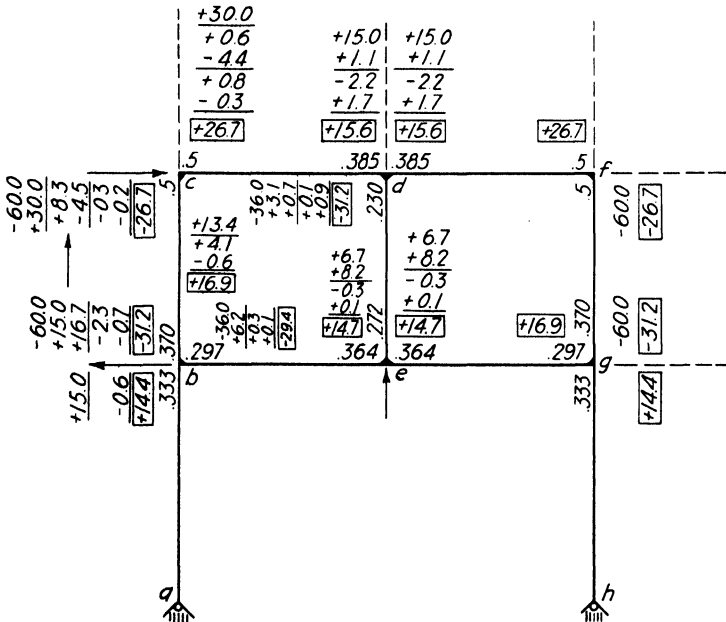



Fig. 6.6b

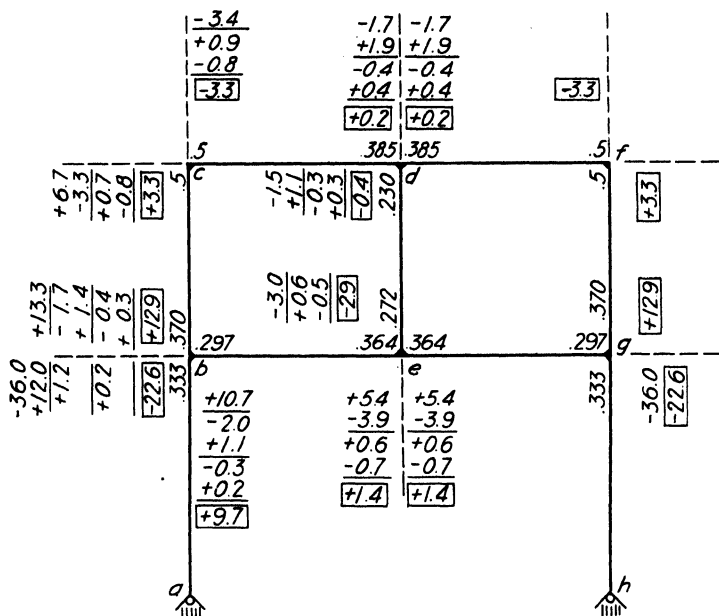



Fig. 6.6c

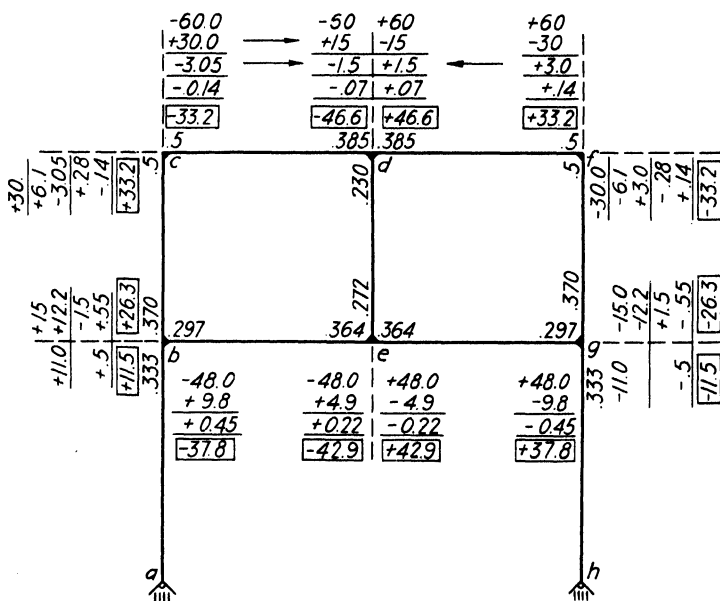
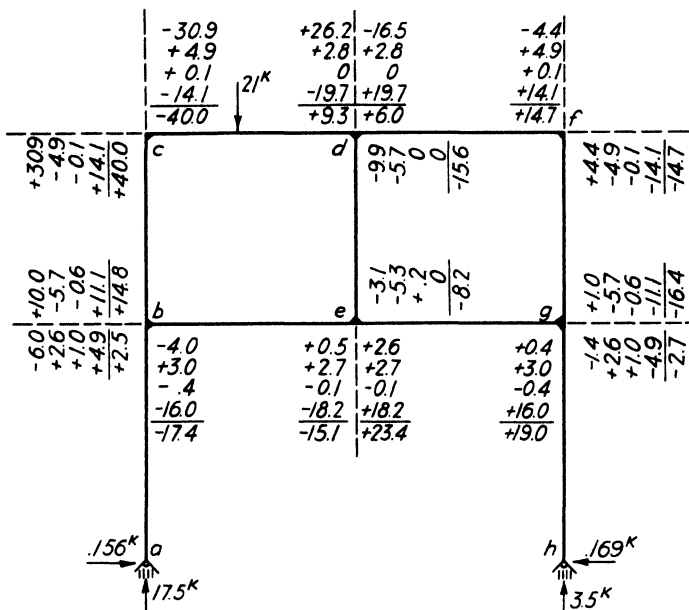



Fig. 6.6d

Fig. 6.6e

The auxiliary forces in terms of Δ_2 are

$$cc' = \frac{2(12.9 + 3.3) - 2.9 - 0.4}{12} = 2.425 \qquad \text{or} \qquad 0.1516\Delta_2 \leftarrow$$

$$bb' = \frac{-22.6 - 22.6}{16} - 2.43 = -5.26 \qquad \text{or} \qquad 0.329\Delta_2 \rightarrow$$

$$ee' = 0$$

For a value of Δ_3 (see Fig. 6.2*d*) equal to 18 the fixed-end moments are

$$M_{Fcd} = M_{Fdc} = -\frac{(6)(10)(18)}{18} = -60, \qquad M_{Fdf} = M_{Ffd} = +60$$

$$M_{Fbe} = M_{Feb} = -\frac{(6)(8)(18)}{18} = -48, \qquad M_{Feg} = M_{Fge} = +48$$

The distribution of these fixed-end moments, as well as the final value of the end moments, is given in Fig. 6.6*d*. The auxiliary forces required to produce these moments are

$$cc' = 0 \qquad bb' = 0$$

$$ee' = \frac{2(33.2 + 46.6 + 37.8 + 42.9)}{18} = 17.84 \qquad \text{or} \qquad 0.99\Delta_3 \downarrow$$

To determine the values of Δ_1, Δ_2, and Δ_3 that will reduce the auxiliary force system to zero, the following equilibrium equations are necessary.

Horizontal forces at c must equal zero, or

$$(a) \qquad 1.225\Delta_1 - 0.1516\Delta_2 - 2.78 = 0$$

Horizontal shear in first story must be zero, or

$$(b) \qquad 1.225\Delta_1 - 1.375\Delta_1 - 0.1516\Delta_2 + 0.329\Delta_2 - 2.78 + 3.24 = 0$$

which gives $-0.150\Delta_1 + 0.1774\Delta_2 = -0.46$.

The discrepancy between $-0.1516\Delta_2$ in (a) and $-0.150\Delta_1$ in (b) is due to approximations in the numerical calculations.

Forces at e must equal zero

$$(c) \qquad -0.99\Delta_3 + 7.54 = 0$$

From (c),

$$\Delta_3 = \frac{7.54}{0.99} = 7.62$$

From (a) and (b),

$$\Delta_1 = 2.18 \qquad \Delta_2 = -0.748$$

The final moments are obtained by multiplying the moments in Fig. 6.6b by 2.18/12, in Fig. 6.6c by $-0.748/16$, and in Fig. 6.6d by 7.62/18, and then by combining these values with the moments of Fig. 6.6a. The sum of the four sets of moments (Fig. 6.6e) gives the actual values.

6.2 *Frames with Inclined Members*

The roof frame of Fig. 6.3a is one example of a continuous frame in which the axes of the various members do not intersect at right angles. Many similar structures can be found in practice, all of which can be analyzed by the preceding methods of solution. The geometrical relations that exist between the linear displacements of the joints for the assumption that the lengths of the members do not change must be always carefully established. Since the construction and application of the Williot diagram for solving this problem are explained in Article 3.8, it will now be used directly in the solution of specific structures.

The bent in Fig. 6.7a, which is a type of structure that is common in viaducts, must undergo the displacements shown by the dotted lines. The relative magnitudes of these displacements in terms of Δ_1 and Δ_2, the horizontal displacements, can be readily obtained from the vector

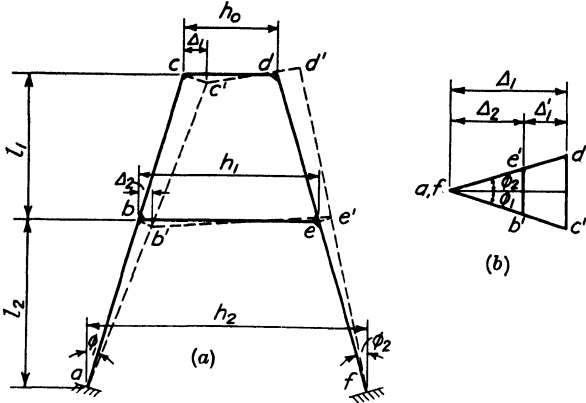



Fig. 6.7

diagram in Fig. 6.7*b*. The rotations of the axes of the various members are the following.

Member *ab*

$$\frac{\Delta_{ab}}{ab} = \frac{\Delta_2}{\cos\phi_1}\frac{\cos\phi_1}{l_2} = \frac{\Delta_2}{l_2}$$

Member *ef*

$$\frac{\Delta_{ef}}{ef} = \frac{\Delta_2}{\cos\phi_2}\frac{\cos\phi_2}{l_2} = \frac{\Delta_2}{l_2}$$

Member *be*

$$\frac{\Delta_{be}}{be} = \frac{\Delta_2(\tan\phi_1 + \tan\phi_2)}{h_1} = \frac{\Delta_2}{l_2}\frac{(h_2 - h_1)}{h_1} \tag{6.4}$$

Member *de*

$$\frac{\Delta_{de}}{de} = \frac{\Delta_1 - \Delta_2}{\cos\phi_2}\frac{\cos\phi_2}{l_1} = \frac{\Delta_1 - \Delta_2}{l_1} = \frac{\Delta_1'}{l_1}$$

Member *cd*

$$\frac{\Delta_{cd}}{cd} = \frac{\Delta_1(\tan\phi_1 + \tan\phi_2)}{h_0} = \frac{(\Delta_1' + \Delta_2)}{l_1}\frac{(h_1 - h_0)}{h_0}$$

The algebraic expressions in equation 6.4 relate to the structural arrangement shown in Figs. 6.7*a* and *b*. The corresponding values for other arrangements should be established from a new displacement diagram.

In the solution of any numerical problem, the unknown moments and forces are taken either in terms of the relative horizontal displacement Δ' in each panel or the actual displacements Δ_1 and Δ_2. The numerical calculations are based on the use of auxiliary force systems as in the preceding problems.

Example 6.3 In the analysis of the frame in Fig. 6.8*a* only half of the structure need be considered in the computations because of symmetry. Thus it is apparent that $\theta_d = \theta_c$ and $\theta_e = \theta_b$, and consequently, the moments at ends of members *cd* and *be* are equal to $6K\theta_c$ and $6K\theta_b$, respectively when only joint rotations are considered. For this condition,

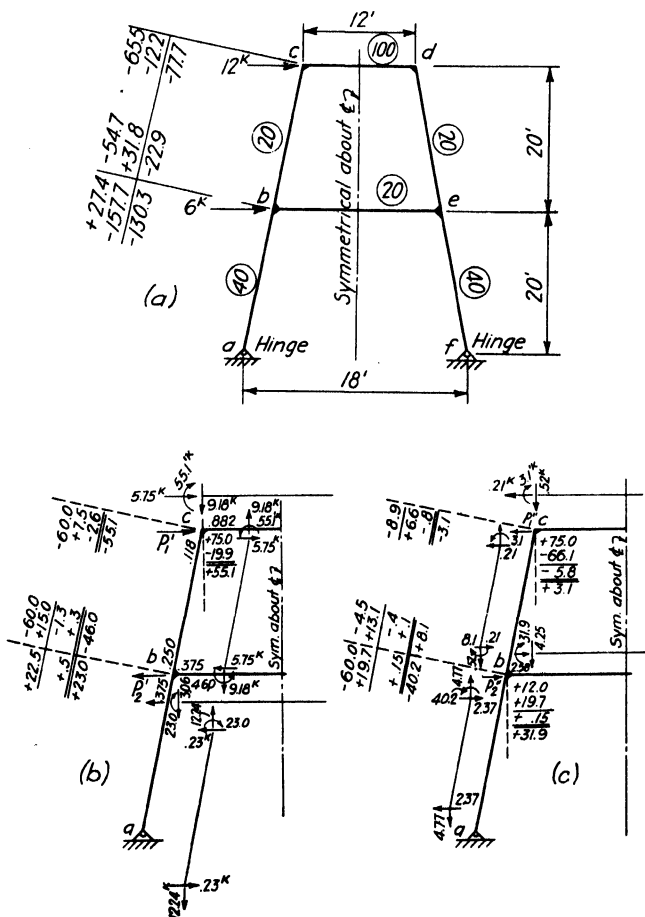



Fig. 6.8

the distribution factors at joints b and c are equal to

Joint b

Member ba	$(3)(40)\theta_b = 120\theta_b$	$r_{ba} = 0.375$
Member be	$(6)(20)\theta_b = 120\theta_b$	$r_{be} = 0.375$
Member bc	$(4)(20)\theta_b = 80\theta_b$	$r_{bc} = 0.250$
	$320\theta_b$	1.000

Joint c

Member bc	$(4)(20)\theta_c = 80\theta_c$	$r_{bc} = 0.118$
Member cd	$(6)(100)\theta_c = 600\theta_c$	$r_{cd} = 0.882$
	$680\theta_c$	1.000

Let us first assume an auxiliary force system P_1' and P_2', together with the reactions, that will produce a relative horizontal movement between joints b and c of Δ_1' equal to 10 units with Δ_2 equal to zero.

For this displacement the fixed-end moments (see equations 6.4) necessary to prevent rotation of the joints are

$$M_{Fba} = 0 \qquad M_{Fbe} = 0 \qquad M_{Fbc} = M_{Fcb} = -\frac{(6)(20)(10)}{20} = -60$$

$$M_{Fcd} = +(6)(100)\frac{(10)}{20}\left(\frac{15-12}{12}\right) = +75$$

The final moments and the horizontal and vertical components of the end forces are given in Fig. 6.8b. It should be noted that the shearing forces in the horizontal members cd and be must be calculated before the vertical and horizontal forces in the columns can be obtained. The values of P_1' and P_2' are equal to

$$P_1' = (2)(5.75) = 11.50 \qquad P_2' = (2)(5.75 + 0.23) = 11.96$$

or, in terms of Δ_1',

$$P_1' = 1.150\Delta_1' \rightarrow, \qquad P_2' = 1.196\Delta_1' \leftarrow$$

In the same manner, the auxiliary force system necessary to give a relative horizontal movement between b and a is determined. Let us assume a value of Δ_2 equal to 10 units and Δ_1' equal to zero. Then

$$M_{Fba} = -\frac{(3)(40)(10)}{20} = -60$$

$$M_{Fbe} = +(6)(20)\frac{10}{20}\frac{3}{15} = +12$$

$$M_{Fcd} = +\frac{(6)(100)(10)}{20}\frac{3}{12} = +75$$

From the moments and forces given in Fig. 6.8*c* the auxiliary force system for any displacement Δ_2 is found to be

$$P_1'' = 0.042\Delta_2 \leftarrow \qquad P_2'' = 0.516\Delta_2 \rightarrow$$

The actual values of Δ_1' and Δ_2 are obtained from the equilibrium conditions that

(*a*)
$$P_1' + P_1'' = 12.0$$
or
$$1.150\Delta_1' - 0.042\Delta_2 = 12.0$$
(*b*)
$$P_2' + P_2'' = 6.0$$
or
$$-1.196\Delta_1' + 0.516\Delta_2 = 6.0$$

from which $\Delta_1' = 11.9$, $\Delta_2 = 39.2$.

The final moments given in Fig. 6.8*a* are obtained by multiplying the moments in Fig. 6.8*b* by 1.19 and those in Fig. 6.8*c* by 3.92.

Example 6.4 The ridge or gable frame illustrated in Fig. 6.9*a* can also be solved by the foregoing procedure with the aid of the geometrical relations derived in Article 3.8 for this type of structure. As the translation of joint *c* is dependent on the movement of joints *b* and *d*, two auxiliary forces, F_b and F_d, will be sufficient to prevent translation of all joints. The fixed-end moments produced by the uniform loads on members *ab* and *bc* are distributed in the usual manner. With the end moments known, the horizontal components at the ends of each member can be

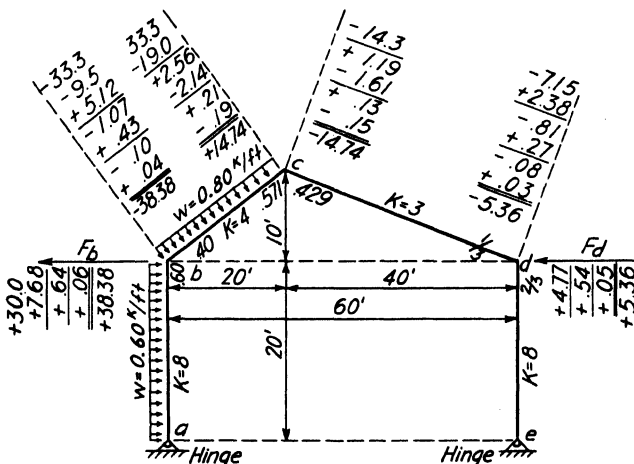



Fig. 6.9a

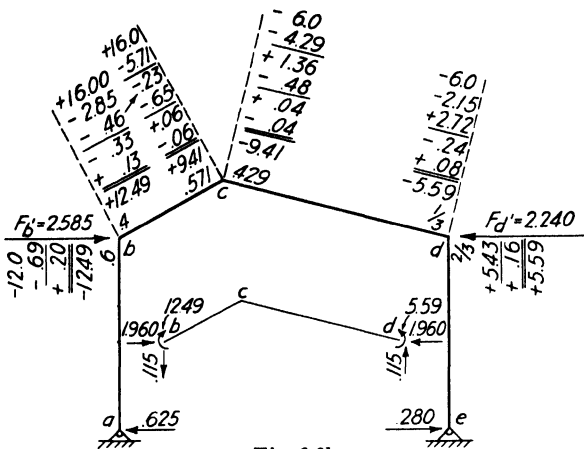



Fig. 6.9b

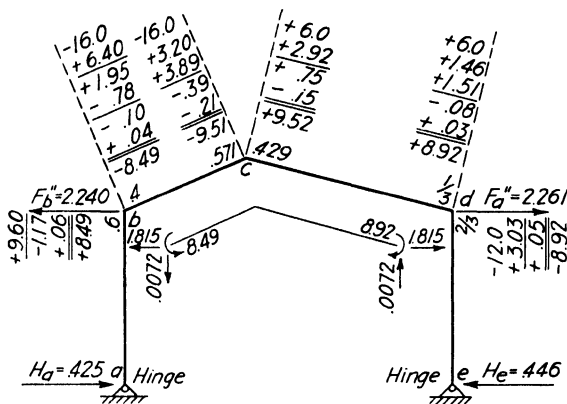



Fig. 6.9c

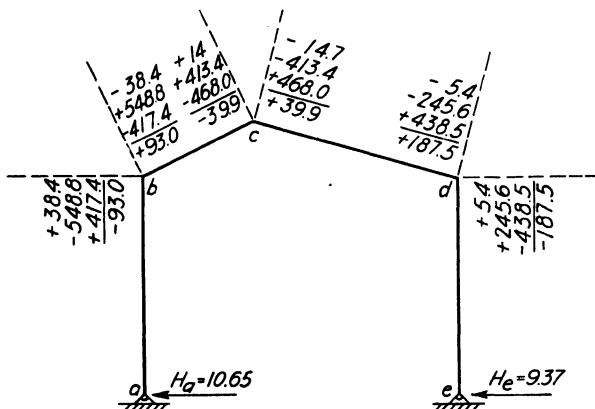



Fig. 6.9d

determined without difficulty. Thus, in members *ab* and *de*, using forces in kips,

$$H_{ba} = \frac{(0.600)(20)(10) + 38.38}{20} = 7.919 \leftarrow H_{de} = \frac{5.36}{20} = 0.268 \leftarrow$$

and, considering the portion *bcd*,

$$V_{dc} = \frac{(16.0)(10) + (8)(5) - 38.38 - 5.36}{60} = 2.604 \uparrow$$

$$V_{bc} = 16.0 - 2.604 = 13.396 \uparrow$$

By taking moments about *c* for forces on member *bc*

$$H_{bc} = \frac{-200 - 38.38 + 14.74 + (13.396)(20)}{10} = 4.428 \rightarrow$$

For member *cd*

$$H_{dc} = \frac{(2.604)(40) + 5.36 + 14.74}{10} = 12.426 \leftarrow$$

The auxiliary forces F_b and F_d will therefore be

$$F_b = 7.919 - 4.428 = 3.491$$

$$F_d = 12.426 + 0.268 = 12.694$$

Forces F_b and F_d must now be removed by superimposing the force systems required to produce the separate displacements Δ_b and Δ_d (see Figs. 6.3*c* and *d*). If joint *b* is assumed to move to the right an amount Δ_b equal to 10 and Δ_d equal to zero, then, by the relations given in equations 3.16 and 3.17 (see Fig. 3.20*b*)

$$\frac{\Delta_b - \Delta_a}{ab} = \frac{10}{20}, \qquad \frac{\Delta_4}{bc} = \frac{\Delta_{bc}}{bc} = \frac{(10)(40)}{(10)(60)} = \frac{2}{3}$$

$$\frac{\Delta_3}{cd} = \frac{\Delta_{cd}}{cd} = \frac{(10)(20)}{(10)(60)} = \frac{1}{3}$$

From these relations the fixed-end moments for the various members for Δ_b equal to 10 are

$$M_{Fba} = -(3)(8)\tfrac{10}{20} = -12.0 \qquad M_{Fbc} = M_{Fcb} = +(6)(4)\tfrac{2}{3} = +16.0$$

$$M_{Fcd} = M_{Fdc} = -(6)(3)\tfrac{1}{3} = -6.0$$

The signs of the fixed-end moments should be determined from the displacement diagram. The correction of these fixed-end moments for rotation of the joints and the values of the auxiliary forces F_b' and F_d' necessary to hold the structure are given in Fig. 6.9*b*.

If joint d is now assumed to move to the right an amount Δ_d equal to 10, whereas joint b is prevented from moving, the fixed-end moments in the various members are

$$M_{Fbc} = M_{Fcb} = -\frac{(6)(4)(10)(40)}{(10)(60)} = -16.0$$

$$M_{Fcd} = M_{Fdc} = +\frac{(6)(3)(10)(20)}{(10)(60)} = +6.0$$

$$M_{Fde} = -(3)(8)\frac{10}{20} = -12.0$$

The moments and auxiliary forces that will produce such a displacement of the frame are given in Fig. 6.9*c*. From the conditions that

$$F_b' + F_b'' + F_b = 0$$

and

$$F_d' + F_d'' + F_d = 0$$

the following equations are obtained:

$$0.2585\Delta_b - 0.2240\Delta_d - 3.491 = 0$$

$$-0.2240\Delta_b + 0.2261\Delta_d - 12.694 = 0$$

from which $\qquad \Delta_b = 439.4, \qquad \Delta_d = 491.6$

The resultant moments (Fig. 6.9*d*) are obtained by combining the moments in Fig. 6.9*a* with 43.94 times those in Fig. 6.9*b* and 49.16 times those in Fig. 6.9*c*.

6.3 *Wind or Earthquake Stresses in Tall Building Frames*

The calculation of stresses in the columns and girders of multiple story building frames that are subjected to shear forces from the action of wind or earthquakes has received considerable attention in technical literature. The theoretically exact solutions provided by strain-energy or slope deflection methods can be applied to this problem but, unless the simultaneous equations are solved by using an electronic computer, the calculations become too complicated for practical design work. In addition, the uncertainty of the actual magnitudes and distribution of the wind or earthquake forces that are carried by any particular assemblage of structural members in a large building tends to discourage the use of complicated analytical methods. It is therefore natural that much attention has been given to more convenient but approximate methods of analysis, several of which are listed in the references at the end of the chapter.

The selection of any particular method of analysis can be criticized so readily that recommendations are probably best avoided. However, among the many proposed simplified solutions, the author believes that the definitely approximate portal method for preliminary design and either the Grinter simplified moment-distribution method or the use of auxiliary restraining forces, as in Example 6.1, for reviewing particular portions of the frame, provides the designer with practical and sufficiently accurate procedures.

The use of auxiliary force systems that establishes the relationship between the shear forces and the relative displacement Δ in each story has already been discussed and will not be repeated here. However, attention will be called to the fact that, since the absolute values of the coefficients of the Δ terms in the shear equations reduce very rapidly for the shear in any particular story, most shear equations will involve only five unknown Δ's regardless of the number of stories. This condition implies that the effect of a relative horizontal displacement Δ in any story will extend to only two stories above and below to any practical degree. In fact, in some cases the use of only one story on each side may be sufficiently accurate.

Therefore, although a twelve story building frame would have twelve unknown Δ values, if the moments in only the bottom four stories are to be reviewed, the relative displacements of the top six stories can be neglected. Furthermore, the numerical values of the Δ terms in the shear equations can be conveniently determined by the iteration procedure. This method is very general in its application. However, for use in the preliminary design and for the reader who is adverse to the solution of simultaneous equations, a brief discussion of the portal method and the simplified moment-distribution method will be presented.

6.4 *Portal Method*

In this method, as presented by Albert Smith in 1915, a building frame such as in Fig. 6.10 is transformed from a highly redundant structure to a statically determinate one by the following assumptions.

1. The total shear H in each story is distributed to the columns in that story by assuming that the shear H_i taken by each inside column is equal to twice the shear H_o taken by each outer column, or, for a frame with n rows of columns

$$2H_o + (n - 2)H_i = 2H_o(n - 1) = H$$

giving

$$H_o = \frac{H}{2(n - 1)} \qquad \text{and} \qquad H_i = 2H_o \tag{6.5}$$

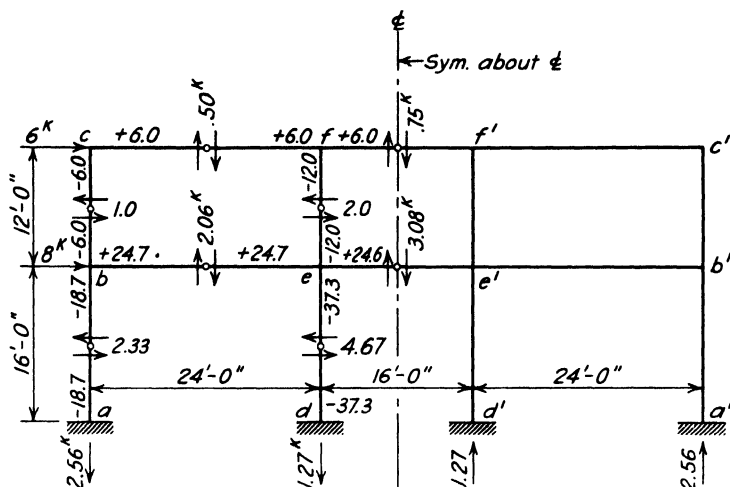



Fig. 6.10

The only basis for this assumption is that the end moments in the interior columns are resisted by two girders, whereas those in the outer columns are resisted by one. Since the actual resistance depends upon the I/L values of both girders and columns as well as the properties of the connections of these members, large variations in shear distribution from the foregoing assumption are possible. However, for a preliminary design of a uniform building frame, the foregoing distribution of shear is reasonable.

2. It is also assumed that a point of contraflexure (zero moment) occurs at the midpoint of each column and each girder. This assumption means that the end moments at the ends of each column are equal in magnitude and are in the same direction. The numerical values of the end moments for the columns in any story of height h are therefore

$$M_o = \frac{H_o h}{2} \qquad \text{and} \qquad M_i = \frac{H_i h}{2}$$

The direction of the end moments is opposite to the direction of the shear couple acting on the column.

The magnitude and direction of the girder end moments are determined from the equilibrium conditions at the joints and the foregoing assumption of a point of contraflexure at midpoint. A numerical example will be given to illustrate the application of these assumptions.

Example 6.5 The shears and end moments in the members of the frame in Fig. 6.10 will be determined by the portal method. In accordance with

the first assumption, the shear H_o in columns ab and $a'b'$ is

$$H_o = \frac{14}{2(4 - 1)} = 2.33 \text{ kips}$$

and in columns de and $d'e'$

$$H_i = (2)(2.33) = 4.67 \text{ kips}$$

From the second assumption the end couples acting on each column are equal to the preceding shears times $h/2$ or 8.0 ft.

For the second story the shear in columns bc and $b'c'$ is

$$H_o = \frac{6}{2(4 - 1)} = 1.0 \text{ kip}$$

and in columns ef and $e'f'$

$$H_i = (2)(1.0) = 2.0 \text{ kips}$$

These columns have end moments equal to the shear times 6.0 ft, which are recorded on the diagram.

After the end moments for all columns are recorded, the end moments for the girders in the two outside bays are then calculated. By taking the summation of all moments at joints b, b', c, and c', respectively equal to zero, we obtain

$$M_{be} = -(-18.7 - 6.0) = +24.7 = M_{eb}$$

and

$$M_{cf} = +6.0 = M_{fc}$$

Likewise, from the summation of moments at joints e, f, e', and f',

$$M_{ee'} = -(-37.3 - 12.0 + 24.7) = +24.6 = M_{e'e}$$

and

$$M_{ff'} = -(-12.0 + 6.0) = +6.0 = M_{f'f}$$

Each girder shear can now be determined by dividing the sum of the end moments by the span length. These values are also tabulated in Fig. 6.10.

All axial forces can now be determined from the equilibrium conditions for the horizontal and vertical forces at each joint. Thus, if joint c is taken first, the axial force in bc is numerically equal and opposite to the shear in cf or 0.5 kip tension.

The axial force in ab is equal to the axial force in bc plus the shear in be or

$$N_{ab} = 0.5 + \frac{(2)(24.7)}{24.0} = 2.56 \text{ kips}$$

In a similar manner,

$$N_{ef} = -0.5 + 0.75 = 0.25 \text{ kip}$$
$$N_{de} = 0.25 - 2.06 + 3.08 = 1.27 \text{ kips}$$

It should be noted that the axial tension or compression in the columns is equal to the accumulation of the girder shears beginning from the top.

A general check on the calculations should be made by taking the summation of the moments of all external forces about any column base. Thus

$$\sum_a M = (6)(28) + (8)(16) - 2(18.7 + 37.3) - (2.56)(64) - (1.27)(40.0 - 24.0) = 0$$

6.5 *Simplified Moment-Distribution Method*

Another procedure for avoiding simultaneous equations in the analysis of building frames was presented by L. E. Grinter (Ref. 4) in 1934. This method is based on the principle that if any selected set of assumed horizontal displacements is used to obtain a corresponding set of fixed-end moments for the columns, the accuracy of the final moments as obtained by the moment-distribution method can be estimated by the variations in the ratios of the actual story shear to the calculated story shear. If it is thought that this variation is too large, a correction of the Δ values or fixed-end moments can be superimposed on the first solution to make the shear ratios more uniform throughout the structure. In effect, the principle involved is to determine by successive corrections an elastic curve that is similar in shape to the actual elastic curve. This method is a practical one, for the final moments and shears are not ordinarily sensitive to small discrepancies in the shear ratios. The amount of work involved will depend, to a considerable degree, on the experience and astuteness of the designer.

Example 6.6 The frame in Fig. 6.10 will be analyzed by the simplified moment-distribution method to illustrate the use of successive corrections. For the first trial solution, a value of $6E\,\Delta/h$ of 10 to the right will be assumed for all columns. The fixed-end moments in the columns in foot-kips is therefore (K-values in Fig. 6.11a)

$$M_{Fab} = M_{Fba} = -\frac{6E\,\Delta}{h}\frac{I}{h} = -10K_{ab} = -80$$
$$M_{Fde} = M_{Fcb} = -(10)(12) = -120$$
$$M_{Fbc} = M_{Fcb} = -(10)(6) = -60$$
$$M_{Fef} = M_{Ffe} = -(10)(10) = -100$$

Since the structure is symmetrical about the center line, similar fixed-end moments will exist in the other columns. The distribution of the fixed-end moments in Fig. 6.11*a* involves no new principles as it is similar to the procedure used in Fig. 6.5. The distributions at the outside joints *b* and *c* are made first followed by *e* and *f*. All corrections due to the rotation of any joint are made before the rotation of the next joint is considered. In this way, both convergence and accuracy are improved. To take advantage of symmetry, a *C* value of 6 was used for members *ee′* and *ff′*. As the *K* values of both girders and columns are fairly uniform, the convergence is quite rapid. If the *K* values of the columns are much larger than those of the girders, the convergence may be very slow.

After the end moments in Fig. 6.11*a* are determined, each story shear can be calculated by dividing the sum of all column end moments in the story by the height *h* of the story, thus

$$H_1' = \frac{2(64.0 + 48.0 + 86.8 + 103.4)}{16} = 37.8$$

$$H_2' = \frac{2(27.6 + 33.3 + 63.4 + 68.2)}{12} = 32.0$$

Fig. 6.11a

-11.2
-3.1
+1.5
-12.8
c .625
-5.6
-6.2
+.8
-.2
-11.2
-13.1
-.5
-13.6
.244 .512
+20.0
-2.1
-6.7
+.7
+.9
+12.8
.375
+33.3
-2.2
-6.2
+.1
-.2
+24.8
.244
f
-12.8
+.7
-12.1
+20.0
-4.3
-3.3
+1.3
+.4
-.1
+14.0
.214
b .500
+33.3
-4.3
-3.1
+.2
+26.7
.152
e .212 .454
Sym. about ℄
-5.7
+1.8
-.2
-4.1
.286
-10.0
-3.0
+3.2
+.2
-.3
-9.9
-5.2
+.3
-4.9
.182
-5.0
-6.0
+1.6
+.3
-9.1
-2.0
a
-2.5
d

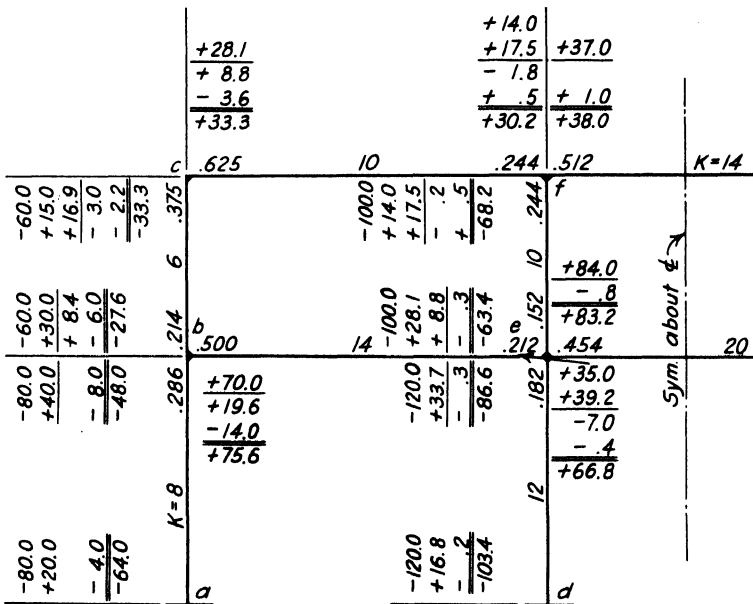



Fig. 6.11b

Fig. 6.11c

Therefore the ratio of the actual shear to the calculated shear for each story is

$$\text{First story} \qquad \frac{14}{37.8} = 0.370$$

$$\text{Second story} \qquad \frac{6}{32.0} = 0.188$$

As these ratios of 0.370 and 0.188 have practically a 100% difference, a correction of the original fixed-end moments should be made. It is apparent that H_1' must be increased and H_2' must be reduced if the two shear ratios are equalized. This condition can be obtained by reducing the fixed-end moments in the second story, which will reduce the shear H_2' in that story and increase H_1' in the first story. As a second trial, the fixed-end moments in the second story will be reduced by $\frac{1}{3}$, that is, additional fixed-end moments of the following amounts will be added by superposition to the first trial solution.

$$M_{Fbc} = M_{Fcb} = +\tfrac{60}{3} = +20.0$$

and

$$M_{Fef} = M_{Ffe} = +\tfrac{100}{3} = +33.3$$

The distribution of these additional fixed-end moments, as shown in Fig. 6.11*b*, is carried out in the same manner as previously explained. Both sets of end moments are given in Fig. 6.11*c*, and from the total, the following new story shears are obtained.

$$H_1' = \frac{2(66.0 + 52.1 + 91.5 + 105.9)}{16} = \frac{631.0}{16} = 39.5$$

$$H_2' = \frac{2(13.6 + 20.5 + 37.3 + 43.4)}{12} = \frac{229.6}{12} = 19.1$$

Therefore the shear ratios for the combined end moments are

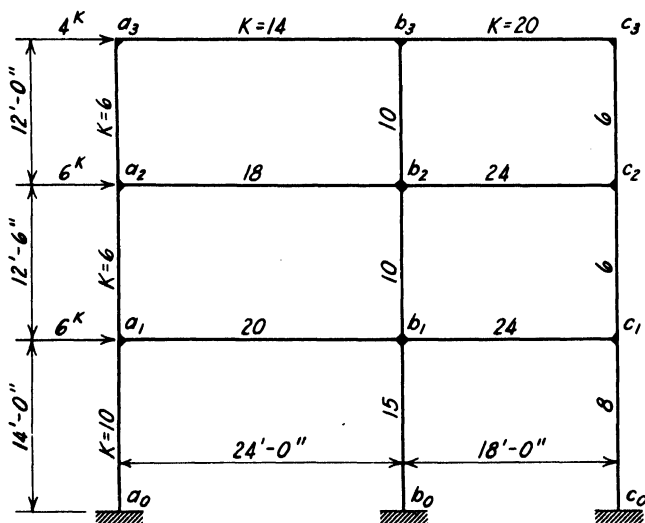
$$\text{First story} \qquad \frac{14}{39.5} = 0.354$$

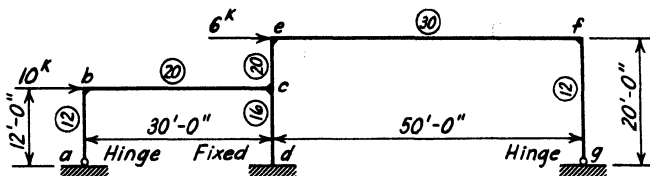
$$\text{Second story} \qquad \frac{6}{19.1} = 0.314$$

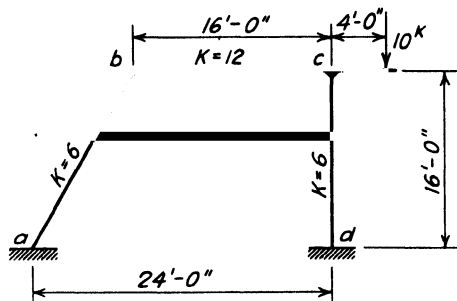
Since these ratios vary by about 12%, the final moments that are determined from them should be quite satisfactory for design purposes. The values of the actual moments are obtained by multiplying the combined values in Fig. 6.11*c* by 0.354 for the first story and by 0.314 for the second story. The numerical results are shown in parentheses in Fig. 6.11*c*. The end moments for the girders at joints *e* and *f* are obtained by distributing the sum of the column moments at those joints in the same proportion as the combined girder moments shown in Fig. 6.11*c*.

Although the moments recorded in Fig. 6.10 which were obtained from the portal method are reasonably close to the values given in Fig. 6.11*c*, no general conclusions as to the accuracy of the portal method should be drawn from this example.

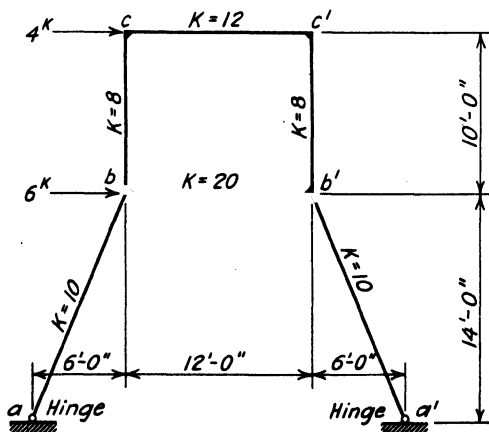
Problems

6.1 Calculate all end moments by the use of relative floor displacements as illustrated in Example 6.1. Check the coefficients of your equations by the reciprocal theorem. Solve the equations by the iteration method.

Problem 6.1

6.2 Determine the end moments for all members of the frame shown: (*a*) by the slope-deflection equations, and (*b*) by the moment-distribution method.

Problem 6.2

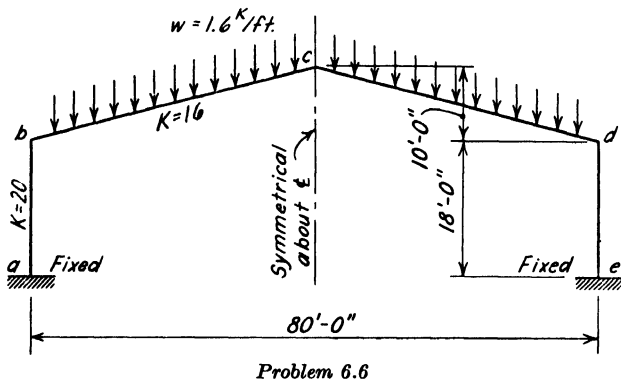
6.3 Solve for the end moments in the frame shown by the moment-distribution method. Construct a displacement diagram assuming no change in length of the members.

Problem 6.3

6.4 (*a*) Calculate all end couples by using auxiliary or restraining joint forces and the moment-distribution method. Construct a displacement diagram for the movement of the joints due to flexure only. Check your equations by the reciprocal theorem.

(*b*) Determine the axial force in members bb' and ab.

Problem 6.4

6.5 Solve Problem 6.1 by the portal method and also by the Grinter simplified moment-distribution method.

6.6 (*a*) Calculate the end moments in the gable frame shown for a uniform vertical roof load of 1.6 kips per foot. Use the moment-distribution method. (*Hint:* Displace joints *b* and *d* simultaneously.)

(*b*) Determine the maximum positive bending moment in the roof member *bc*.

Problem 6.6

6.7 Determine the reactions in the gable frame of Problem 6.6 for a temperature drop of 60°F. Use a coefficient of linear expansion of 6.5×10^{-6} per °F and assume that the absolute value of EI/L is equal to $4 \times 10^4 \times K$ in.-kips where K is the relative value in the diagram.

6.8 Solve Problem 6.6 for hinged bases.

PANEL METHOD FOR ANALYZING QUADRANGULAR FRAMES

6.6 *Historical Development*

In 1904, Professor L. F. Nicolai presented a solution for a parallel-chord Vierendeel truss based on the assumption that the rectangular panel, composed of the upper and lower chords and the two verticals, is the fundamental structural unit and that the continuity with the remainder of the structure can be ignored.

Nicolai's analysis was then made by assuming that diagonals are acting in the panels and that the members are pin-connected as shown in Fig. 6.12*a*. Then at each joint an external force is applied (Fig. 6.12*b*) that will neutralize the stress in the diagonal. The last step is to consider the panel as a rigid frame acted upon by these forces, as, for example, panel *abcd*, Fig. 6.12*c*. The moments in this frame are assumed to be equal to the moments in the Vierendeel truss. Obviously, such a procedure gives only an approximate solution, for the continuity between panels is ignored.

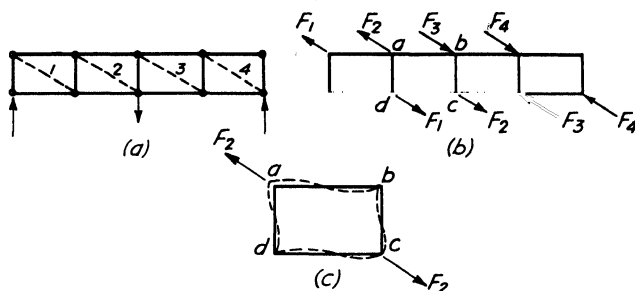



Fig. 6.12

This method, used by Professor Nicolai, was greatly extended and made more practicable in 1921 by K. Čališev, who showed that the verticals of a particular panel could be increased in stiffness until they provided enough additional restraint to compensate for the remainder of the structure. This correction took the form of a rapidly converging series, and therefore changed the method from an approximate solution to a solution by successive approximations. Čališev also extended the solution to Vierendeel trusses with inclined chords. However, he still retained the idea of reducing the diagonal stress to zero, which is an unnecessary complication, for the forces acting on the panel can be determined without considering imaginary diagonal members at all. In the following derivation, the author has therefore modified the procedure by considering the forces acting directly on the panel and by correcting the moments for continuity between the panels rather than by correcting the stiffness of the verticals. These changes make the method analogous to the moment-distribution method except that the action of the panel rather than the joint is the primary factor. The use of the panel permits both rotation and translation of the joints to occur simultaneously and still give a solution by successive approximations that will converge rapidly. The author recommends the use of the panel method for Vierendeel truss systems whenever the chords have approximately the same I/L value, say with a difference less than 15%.

6.7 *Forces Acting on a Panel*

If any panel *abcd* of a continuous frame of the Vierendeel truss type, Fig. 6.13*a*, in which the chord members *ad* and *bc* have the same K or I/L value, is separated from the structure by passing sections 1-1 and 2-2, the force system acting on the panel will be as shown in Fig. 6.13*b*. In representing these forces and in the subsequent analysis, the following

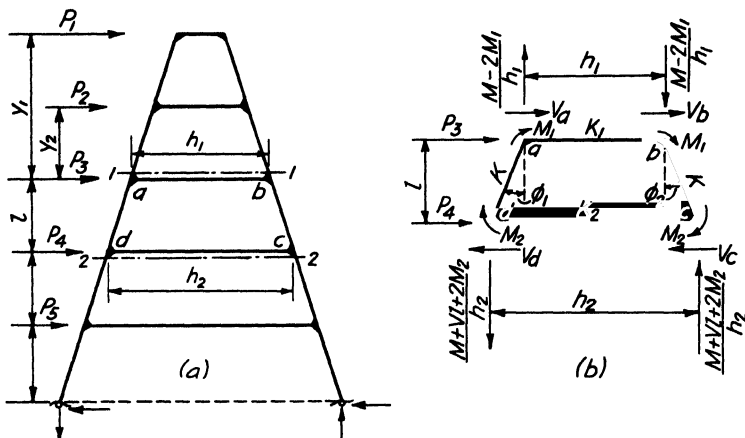



Fig. 6.13 Forces acting on a panel of a quadrangular frame.

notation is used:

M = bending moment on section 1-1 = $P_1y_1 + P_2y_2$.

V = shear in panel *abcd* = $P_1 + P_2 + P_3$.

l = panel length

$K = \dfrac{I}{L}$ value of chord members *ad* and *bc*.

$K_1, K_2 = \dfrac{I}{L}$ values of members *ab* and *cd*.

$$r = \frac{K}{K_1}, \qquad s = \frac{K}{K_2}, \qquad \alpha = \frac{h_2 - h_1}{h_1}$$

$$D = 6 + r + s + \alpha(2\alpha + \alpha s + 2s + 6).$$

For purposes of analyzing the moments in the panel *abcd*, Fig. 6.13*b*, the actual force system can be resolved into three equivalent force

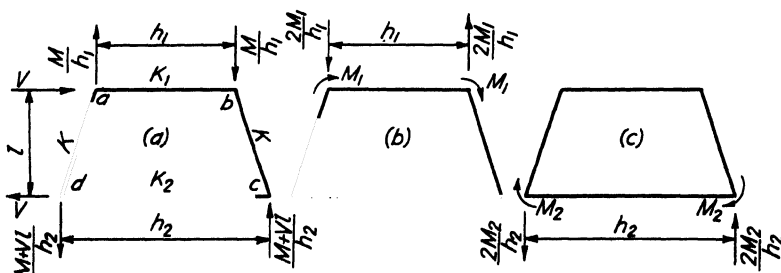



Fig. 6.14 Primary and secondary force systems acting on a panel.

systems. These are shown in Figs. 6.14*a*, *b*, and *c*, and by inspection it can be seen that they add up to the original system.

6.8 *Primary Moments*

The force system shown in Fig. 6.14*a* represents the action of the external forces, and consequently the moments produced by these forces are necessary for structural stability. Therefore, as this force system represents the principal structural action, the resulting moments will be called the *primary moments*. The values of the primary moments are given by the following equations:

$$
\begin{aligned}
M_{ad}' = M_{bc}' &= \frac{\alpha M - Vl}{2D}[3 + s + \alpha(2 + s)] \\
M_{da}' = M_{cb}' &= \frac{\alpha M - Vl}{2D}(3 + r + \alpha)
\end{aligned} \qquad (6.6)
$$

The equations for the primary moments may be obtained by solving for the moments in Fig. 6.15*a* by the slope deflection equations. From the displacement diagram in Fig. 6.15*b* all translations can be expressed in terms of Δ, the horizontal movement of joints *a* and *b*. These values are

$$
\frac{\Delta_{ad}}{ad} = \frac{\Delta}{\cos\phi_1}\frac{\cos\phi_1}{l} = \frac{\Delta}{l} = R \qquad \frac{\Delta_{bc}}{bc} = \frac{\Delta}{\cos\phi_2}\frac{\cos\phi_2}{l} = \frac{\Delta}{l} = R
$$

$$
\frac{\Delta_{ab}}{h_1} = \Delta\frac{(\tan\phi_1 + \tan\phi_2)}{h_1} = \frac{\Delta}{l}\frac{h_2 - h_1}{h_1} = \alpha R
$$

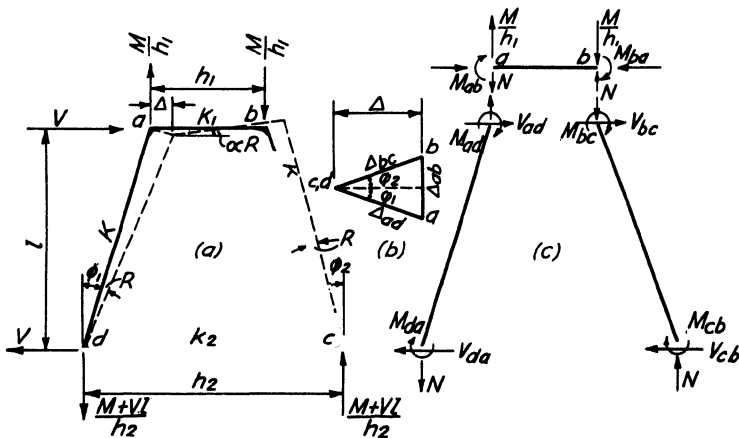



Fig. 6.15 Internal forces due to the primary force system.

It can be shown that $\theta_a = \theta_b$ and $\theta_d = \theta_c$ regardless of the values of the angles ϕ_1 and ϕ_2. Therefore the slope deflection equations for the end moments will be (dropping the primes during the derivation)

$$M_{ab} = M_{ba} = 6K_1\theta_a + 6K_1\alpha R$$
$$M_{ad} = M_{bc} = 4K\theta_a + 2K\theta_d - 6KR$$
$$M_{da} = M_{cb} = 2K\theta_a + 4K\theta_d - 6KR$$
$$M_{dc} = M_{cd} = 6K_2\theta_d$$

The equilibrium equations are

$$(a)\quad M_{ab} + M_{ad} = 0 \qquad (b)\quad M_{da} + M_{dc} = 0$$

and, from equilibrium of members *ad* and *bc* (Fig. 6.15*c*) ($\Sigma M = 0$),

$$M_{ad} + M_{da} + M_{bc} + M_{cb} - N(h_2 - h_1) + Vl = 0$$

The equilibrium of member *ab* gives ($\Sigma M_b = 0$)

$$N = \frac{M_{ab} + M_{ba} + M}{h_1}$$

which make the third equilibrium equation equal to

$$(c)\quad M_{ad} + M_{da} + M_{bc} + M_{cb} - (M_{ab} + M_{ba} + M)\frac{h_2 - h_1}{h_1} + Vl = 0$$

By substituting the slope deflection equations in the equilibrium equations (*a*), (*b*), and (*c*) and calling

$$\frac{K}{K_1} = r \qquad \frac{K}{K_2} = s \qquad \frac{h_2 - h_1}{h_1} = \alpha$$

we obtain

$$(a')\quad (2r + 3)\theta_a + r\theta_d - 3(r - \alpha)R = 0$$
$$(b')\quad s\theta_a + (2s + 3)\theta_d - 3sR = 0$$
$$(c')\quad (r - \alpha)\theta_a + r\theta_d - (2r + \alpha^2)R = \frac{\alpha M - Vl}{12K_1}$$

The solution of these equations gives the results

$$\theta_a = \frac{\alpha M - Vl}{12K_1}\left(\frac{-rs - 3r + 2\alpha s + 3\alpha}{rD}\right)$$
$$\theta_d = \frac{\alpha M - Vl}{12K_1}\left(\frac{-rs - 3s - \alpha s}{rD}\right) \qquad (6.7)$$
$$R = \frac{\alpha M - Vl}{12K_1}\left(\frac{-rs - 2r - 2s - 3}{rD}\right)$$

in which $D = 6 + r + s + \alpha(2\alpha + \alpha s + 2s + 6)$.

When these values of θ_a, θ_d, and R are substituted back into the slope deflection equations, the primary moments of equation 6.6 are obtained.

By equation 6.6, the primary moments M' are easily computed for each panel and recorded on a sketch of the structure. The external moments M and Vl are considered positive when they act clockwise on the panel. The internal moments M' that act on the members *ad* and *bc* are also clockwise if positive. The value of α can be either positive or negative.

6.9 *Secondary Moments*

The force systems shown in Figs. 6.14*b* and *c* represent the effect of the internal moments in the adjacent panels and therefore exist as a result of continuity of the panels. Since the moments M'' and M''' produced by these two force systems are not necessary for structural stability in the panel *abcd*, they will be defined as secondary moments. The magnitude of these secondary moments can be computed without difficulty by the following equations, which are derived in the same manner as equation 6.6.

$$M_{ad}'' = M_{bc}'' = +\frac{r}{D} M_1 \tag{6.8a}$$

$$M_{da}'' = M_{cb}'' = -\frac{r(1+\alpha)}{D} M_1 \tag{6.8b}$$

$$M_{ad}''' = M_{bc}''' = -\frac{s(1+\alpha)}{D} M_2 \tag{6.9a}$$

$$M_{da}''' = M_{cb}''' = +\frac{s(1+\alpha)^2}{D} M_2 \tag{6.9b}$$

Evidently the constants r/D, $r(1+\alpha)/D$, $s(1+\alpha)/D$, and $s(1+\alpha)^2/D$ are correction factors that can be computed for each panel and recorded on a sketch of the structure. The primary moments M' in the adjacent panels can be used for the first approximation of M_1 and M_2, and the corrections can then be computed as for any method of successive approximations. A convenient numerical arrangement for making the calculations is as follows:

1. Compute and tabulate r, s, α, D, r/D, $r(1+\alpha)/D$, $s(1+\alpha)/D$, and $s(1+\alpha)^2/D$ for each panel.
2. Compute the primary moments by equation 6.6, and record on a sketch of the frame.
3. Make the necessary corrections by equations 6.8 and 6.9, using the recorded correction factors.

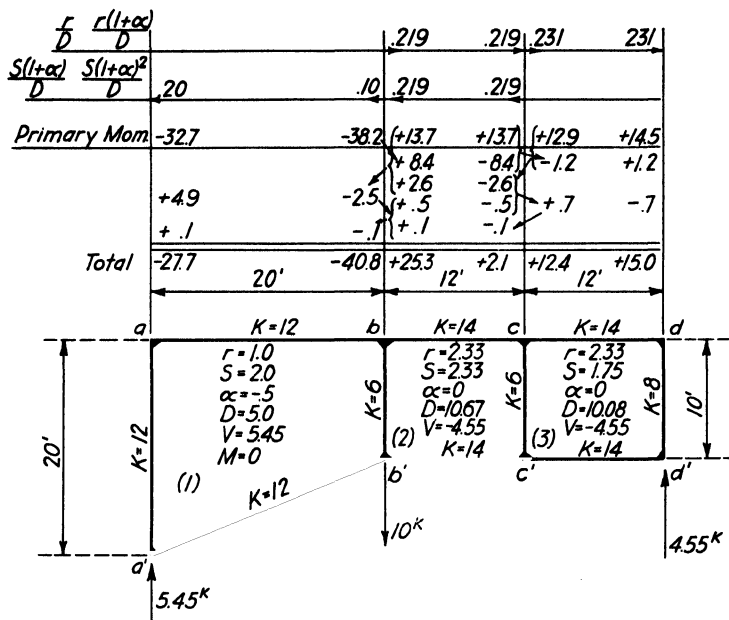



Fig. 6.16

4. Determine the sign of any secondary moment acting on the chord members in a panel directly from the sign of the moment in the adjacent panel producing it by the following rule. Adjacent secondary moments have a sign opposite to that of the applied chord moment ($-M_1$ or $-M_2$), whereas secondary moments at the far end of the panel have the same sign as the applied chord moment. This sign convention assumes that the recorded moments act on the chord members and not on the joints.

Example 6.7 The moments in the frame of Fig. 6.16 will be computed by the panel method. The calculation of the various constants that are recorded on the sketch requires no special explanation. The primary moments in foot-kips as given by equation 6.6 are the following.

PANEL 1

$$M_{ab}' = \frac{(-0.5)(0) - (5.45)(20)}{(2)(5.0)}[3 + 2.0 - (0.5)(2 + 2.0)] = -32.7$$

$$M_{ba}' = \frac{(-0.5)(0) - (5.45)(20)}{(2)(5.0)}(3 + 1.0 - 0.5) = -38.2$$

The moments and shears are taken from the left; therefore

$$M = 0 \qquad V = 5.45 \qquad h_1 = 20 \qquad h_2 = 10$$

PANEL 2

$$M_{bc}' = \frac{-(-4.55)(12)}{(2)(10.67)}(3 + 2.33) = +13.7 \quad \text{both ends}$$

PANEL 3

$$M_{cd}' = \frac{-(-4.55)(12)}{(2)(10.08)}(3 + 1.75) = +12.9$$

$$M_{dc}' = \frac{-(-4.55)(12)}{(2)(10.08)}(3 + 2.33) = +14.5$$

The corrections are then made in the order indicated by the arrows (Fig. 6.16); that is, panel 2 was corrected for the moment −38.2 occurring in panel 1, the absolute values of the corrections being (0.219)(38.2) at either end. To determine the sign of the correction, the rule previously stated can be used, that is, the correction at the far end of the panel has the same sign as, and that at the near end has a sign opposite to, the recorded chord moment producing the correction. Therefore $\Delta M_{cb} = -8.4$ and $\Delta M_{bc} = +8.4$. The corrections in panel 3 due to the moments $(+13.7 - 8.4) = +5.3$ acting on the left side will have a plus sign (+1.2) on the far end and a minus sign (−1.2) at the adjacent end. The corrections in panel 1 due to $(+13.7 + 8.4 + 2.6) = +24.7$ will have a plus sign (+4.9) at the far end and a minus sign (−2.5) at the adjacent end. After the corrections have been carried out to the desired accuracy, the primary and secondary moments are added algebraically to give the actual value.

6.10 *Frames Fixed at the Base*

If the frame has columns that are fixed at the base, Fig. 6.17, the solution can be made by assuming a member *dc*, shown by the dotted line, with $K_2 = \infty$. For this panel, since $s = K/K_2 = 0$, the primary and secondary moments can be computed without difficulty.

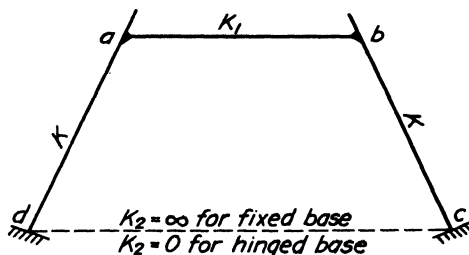



Fig. 6.17 Equivalent members for fixed and hinged bases.

6.11 *Frames Hinged at the Base*

If the frame is hinged at c and d (Fig. 6.17), an imaginary member dc with $K_2 = 0$ can be used. This condition means that $s = K/K_2 = \infty$. For this case the primary moments become

$$M_{ad}' = M_{bc}' = \frac{\alpha M - Vl}{2} \lim_{s\to\infty} \frac{[(3 + 2\alpha)/s] + 1 + \alpha}{\dfrac{6 + r + 2\alpha^2 + 6\alpha}{s} + 1 + 2\alpha + \alpha^2}$$

$$= \frac{\alpha M - Vl}{2}\left(\frac{1}{1 + \alpha}\right) \tag{6.10a}$$

$$M_{da}' = M_{cb}' = \frac{\alpha M - Vl}{2} \lim_{s\to\infty} \frac{(3 + r + \alpha)/s}{\dfrac{6 + r + 2\alpha^2 + 6\alpha}{s} + 1 + 2\alpha + \alpha^2} = 0 \tag{6.10b}$$

The quantities in equation 6.10 were obtained by dividing the numerator and denominator of equation 6.6 by s. By this procedure, it can be easily seen that the secondary moment due to M_1 will be zero as s approaches ∞.

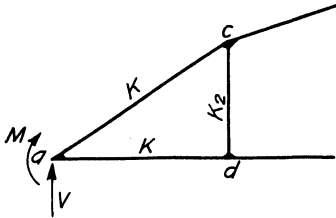



Fig. 6.18 Illustration of a triangular panel.

6.12 *Moments in Triangular Panels* ($\alpha = \infty$)

In many Vierendeel trusses, it is desirable to use triangular end panels as shown in Fig. 6.18. For this case $h_1 = 0$ and $\alpha = (h_2 - h_1)/h_1 = \infty$. The primary moments will then be

$$M_{ac}' = M_{ad}'$$

$$= \lim_{\alpha\to\infty} \frac{(M/2)[(3 + s)/\alpha + 2 + s] - (Vl/2)[(3 + s)/\alpha^2 + (2 + s)/\alpha]}{\dfrac{6 + r + s}{\alpha^2} + \dfrac{2s + 6}{\alpha} + 2 + s} = \frac{M}{2} \tag{6.11a}$$

$$M_{ca}' = M_{da}'$$

$$= \lim_{\alpha\to\infty} \frac{(M/2)[(3 + r)/\alpha + 1] - (Vl/2)[(3 + r)/\alpha^2 + 1/\alpha]}{\dfrac{6 + r + s}{\alpha^2} + \dfrac{2s + 6}{\alpha} + 2 + s} = \frac{M}{2}\left(\frac{1}{2 + s}\right) \tag{6.11b}$$

The values in equation 6.11 were obtained by dividing numerator and denominator of equation 6.6 by α^2.

The secondary moments for $\alpha = \infty$ will be

$$M'''_{ac} = M'''_{ad} = \lim_{\alpha \to \infty} \frac{s(1 + \alpha)/\alpha^2}{D/\alpha^2} M_2 = 0 \tag{6.12a}$$

$$M'''_{ca} = M'''_{da} = \lim_{\alpha \to \infty} \frac{s[(1/\alpha) + 1]^2}{D/\alpha^2} M_2 = \left(\frac{s}{2 + s}\right) M_2 \tag{6.12b}$$

Example 6.8 The end moments acting on all members of the bent in Fig. 6.4, Example 6.1, will be calculated by the panel method. In this structure, Fig. 6.19, the chord members (columns) of the panels are parallel and therefore all α values are zero.

The constants and primary moments for the various panels are the following.

PANEL 1 (FIRST STORY)

$$r = \tfrac{12}{14} = 0.86 \qquad s = \tfrac{12}{0} = \infty \qquad \alpha = 0$$

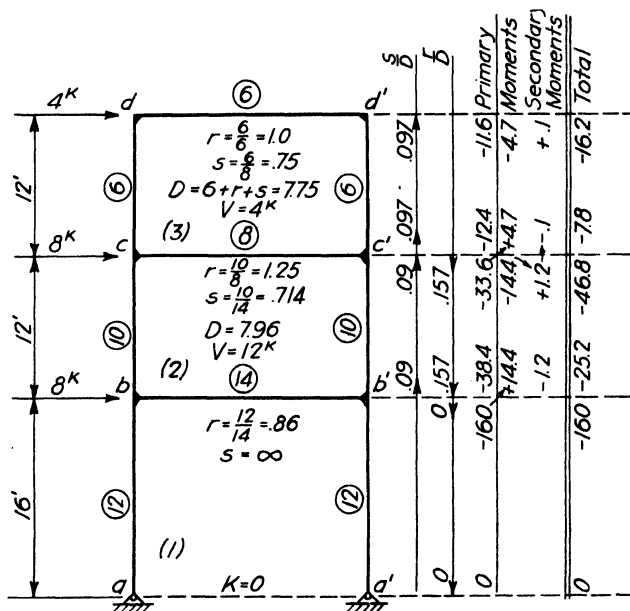



Fig. 6.19 Frame bent analyzed by the panel method.

From Article 6.11, equations 6.10*a* and *b*,

$$M_{ab}' = M'_{a'b'} = 0$$

$$M_{ba}' = M'_{b'a'} = -\frac{Vl}{2} = -\frac{(20)(16)}{2} = -160 \text{ ft-kips}$$

PANEL 2 (SECOND STORY)

$$r = \tfrac{10}{8} = 1.25 \qquad s = \tfrac{10}{14} = 0.714 \qquad D = 6 + r + s = 7.96$$

From equation 6.6, $\alpha = 0$

$$M_{bc}' = -\frac{Vl}{2D}(3 + r) = -\frac{(12)(12)}{(2)(7.96)}(4.25) = -38.4$$

$$M_{cb}' = -\frac{Vl}{2D}(3 + s) = -\frac{(12)(12)}{(2)(7.96)}(3.71) = -33.6$$

PANEL 3 (THIRD STORY)

$$r = \tfrac{6}{6} = 1.0 \qquad s = \tfrac{6}{8} = 0.75 \qquad D = 7.75$$

$$M_{cd}' = -\frac{(4)(12)}{(2)(7.75)}(3 + 1.0) = -12.4$$

$$M_{dc}' = -\frac{(4)(12)}{(2)(7.75)}(3 + 0.75) = -11.6$$

The correction factors for the calculations of the secondary moments due to continuity of the panels are the following.

PANEL 1

$$\frac{r}{D} = \frac{r}{6 + r + s} = \frac{r/s}{(6 + r)/s + 1} = 0 \qquad \text{when } s = \infty$$

Therefore there is no correction in the first story.

PANEL 2

$$\frac{s}{D} = \frac{0.714}{7.96} = 0.09$$

$$\frac{r}{D} = \frac{1.25}{7.96} \doteq 0.157$$

PANEL 3

$$\frac{s}{D} = \frac{0.75}{7.75} = 0.097$$

The primary moments and the correction factors are recorded on the diagram in Fig. 6.19, and the corrections are then made as indicated by the arrows. The moments in panel 2 that are caused by the end moment

-160 in panel 1 should be calculated first, for these are the largest corrections. These moments are

$$M_{bc}'' = -(-160)\frac{s}{D} = (160)(0.09) = 14.4$$

$$M_{cb}'' = (-160)\frac{s}{D} = -14.4$$

The correction at the opposite end of the panel is always the same sign as the recorded chord moment producing it; the adjacent correction is of opposite sign.

After these corrections are added to the primary moments in panel 2, panel 3 is corrected for the total effect, that is, $-33.6 - 14.4$, or -48.0. These corrections are

$$M_{cd}'' = -(-48.0)(0.097) = 4.7$$

$$M_{dc}'' = -(48.0)(0.097) = -4.7$$

Panel 2 is then corrected for the total moment in panel 3, that is, for $-12.4 + 4.7$, or -7.7. These corrections are

$$M_{bc}''' = (-7.7)\frac{r}{D} = (-7.7)(0.157) = -1.2$$

$$M_{cb}''' = -(-7.7)(0.157) = +1.2$$

Panel 3 is then corrected for the additional moment $+1.2$ in panel 2, or

$$M_{cd}'' = -(1.2)(0.097) = -0.12$$

$$M_{dc}'' = (1.2)(0.097) = +0.12$$

Any additional corrections are too small to consider. The final moments check the values obtained in Example 6.1.

Example 6.9 The panel method is particularly convenient for the analysis of inclined chord Vierendeel trusses such as the viaduct bent in Fig. 6.20. The numerical values of the various coefficients are tabulated on the diagram for each panel. The primary moments and the coefficients for determining the secondary moments will be calculated for each panel.

PANEL 1. As the structure is hinged at bases a and a', the primary moment M_{ba}' is given by equation 6.10a.

$$M_{ba}' = \frac{(0.173)(192) - (16)(12)}{2}\left(\frac{1}{1 + 0.173}\right) = -67.7 \text{ ft-kips}$$

Fig. 6.20

Because of symmetry, this value can be determined directly from the horizontal and vertical components of the reaction at a, or

$$M_{ba}' + (8)(12) - (16)(1.76) = 0$$

giving

$$M_{ba}' = -96 + 28.3 = -67.7 \text{ ft-kips}$$

As stated in Article 6.11, the secondary moments in panel 1 produced by any adjacent moments in panel 2 as given by equation 6.8 are zero since $s = \infty$.

PANEL 2. By the use of equation 6.6 the primary moments M_{cb}' and M_{bc}' are readily determined as follows:

$$M_{cb}' = \frac{(0.168)(72) - (12)(10)}{(2)(10.61)}[3 + 1.5 + (0.168)(3.5)] = -25.8$$

$$M_{bc}' = -5.08(3 + 1.5 + 0.168) = --23.8$$

The absolute values of the correction factors for the effect of the moments in panel 3 are r/D and $r(1 + \alpha)/D$ or 0.141 and 0.165. The correction factors for the effect of the moments in panel 1 are $s(1 + \alpha)/D$

and $s(1 + \alpha)^2/D$ or 0.165 and 0.193. The sign of the correction should be obtained from the rule already explained in Article 6.9 and Example 6.7.

PANEL 3. The primary moments M_{dc}' and M_{cd}' are

$$M_{dc}' = \left[\frac{-(6)(12)}{(2)(9.41)}\right][3 + 1.0 + (0.252)(3.0)] = -18.2$$

$$M_{cd}' = -3.83(3 + 0.2 + 0.252) = -13.2$$

The absolute values of the correction factors for the effect of moments in panel 2 are $s(1 + \alpha)/D$ and $s(1 + \alpha)^2/D$ or 0.133 and 0.166.

The four corrections to the primary moments have been made in the order of panels 1 to 2, 2 to 3, 3 to 2, and finally 2 to 3. Once the physical meaning of the operations is recognized, the best arrangement of the corrections is usually apparent.

6.13 *Effect of Panel Proportions*

The term "panel proportions" refers here to the height of the verticals and the length of panels, but does not include the stiffness of the various members. By equation 6.6 which neglects axial deformation, it can be seen that the primary moments, and therefore the total moments in any panel, will be zero if $\alpha M - Vl$ is equal to zero. This condition will be satisfied whenever the ratio α, that is, $(h_2 - h_1)/h_1$ is made equal to Vl/M for the system of applied loads. If the applied load is uniformly distributed over the span, as it usually is for the dead weight, the equilibrium polygon is a parabola, and therefore a parabolic curve for the top or bottom chords will be most economical. The most uneconomical Vierendeel truss will be one with parallel chords. The variation in the end moments acting on the chords when the axis of the top chord is moved above or below a parabola can be readily seen from the influence diagrams in Fig. 6.21. For truss A, in which the joints of the top chord lie on a parabola, the positive and negative areas of the influence lines are equal, whereas for trusses B and C, which lie above and below a parabola, the areas are unequal.

Fig. 6.21 Influence lines for moments in Vierendeel trusses.

The total combined dead and live load moments for the three different trusses of Fig. 6.21 are compared in Table 6.2. These moments were calculated from the influence diagrams for a dead load of 1500 lb per

linear foot of truss and for the standard H-20 truck loading plus 30% impact. These values indicate that the maximum moments at certain sections for trusses *B* and *C* are 50 to 60% higher than the corresponding moments for truss *A*. With a reduced dead load and a larger concentrated live load, truss *B* may give a more favorable comparison, but truss *C* is unsatisfactory for any practical loading.

Table 6.2 Summary of Moments for Dead Load of 1500 Pounds per Foot of Truss and American Standard H-20 Highway Loading

Moment	Truss	Dead Load	Live Load + 30% Impact		Maximum Combined (ft-kips)
			+	−	
	A	0	140.5	73.0	+140.5
M_{bc}	*B*	−48.7	125.0	97.8	−146.5
	C	+37.5	154.0	53.2	+191.5
	A	0	70.8	69.4	+70.8
M_{cb}	*B*	−25.3	66.1	82.3	−107.6
	C	+5.8	71.5	64.5	+77.3
	A	0	122.5	94.1	+122.5
M_{cd}	*B*	−17.6	113.8	101.6	−119.2
	C	+45.8	147.2	75.8	+193.0
	A	0	87.5	93.6	−93.6
M_{dc}	*B*	−35.0	72.5	115.3	−150.3
	C	+68.3	112.1	53.6	+180.4

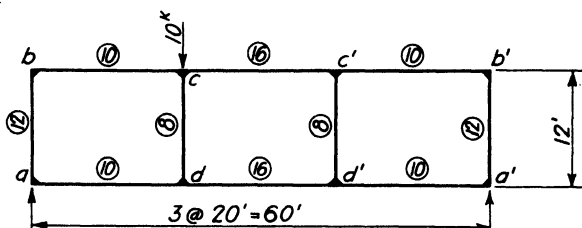
This comparison has been obtained on the assumption that the lengths of the members do not change. The error due to this assumption is relatively small except for shallow parallel chord trusses with long panels. For parabolic chord trusses the effect of axial stress on the magnitude of the end moments can be neglected unless the truss is unusually shallow. The moments caused by translation of the joints produced by the change in length of the members can be ascertained in the same manner as the so-called secondary stresses in triangular trusses. This problem is treated in the following section.

The parabolic chord Vierendeel truss is somewhat similar to the parabolic tied arch that is frequently constructed in the United States. The principal difference between these two types of structures is in the distribution of the bending moments. In an arch with a flexible tie

member, the bending moments are taken entirely by the arch rib, whereas in the Vierendeel truss they are distributed between the upper and lower chord members and the verticals. The total amount of material required for each structure is about the same.

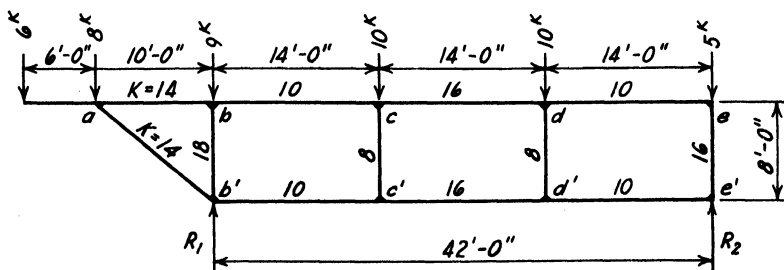
Problems

6.9 Compute the moments in the Vierendeel truss shown. Note that three relative vertical displacements can be used as shown by a displacement diagram that is started with member *ab* assumed vertical; or two vertical displacements and one horizontal displacement can be used if points *a* and *a′* are kept at the same elevation. The choice lies in whether to use the Williot diagram direct or to correct for rotation. The end moments are the same in either case.

Ans. $M_{bc} = -30.9$ ft-kips; $M_{cb} = -35.8$ ft-kips; $M_{cc'} = +26.9$ ft-kips.

Problem 6.9

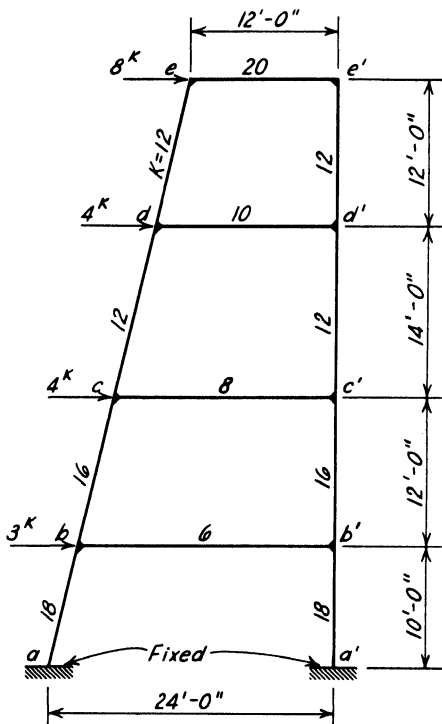
6.10 Solve Problem 6.3 by using the panel method to determine the end moments resulting from removing the auxiliary force at *b* that was necessary to prevent sidesway.

6.11 Calculate the end moments for all members of the frame in Problem 6.4 by the panel method.

6.12 (*a*) Calculate all end moments for the frame shown and tabulate the values on a diagram.

Problem 6.12

(*b*) What are the values of the axial components in members *ab′* and *bb′*?

(*c*) Show the horizontal and vertical components for all members at joint *b′* and check the equilibrium conditions.

6.13 Calculate the end moments in all members and record the values on a sketch.

Problem 6.13

SECONDARY STRESSES IN TRIANGULAR TRUSSES

6.14 Introduction

In the analysis of riveted or welded triangular trusses, the axial stresses are first computed on the assumption that the ends of the members are pin-connected. Such an assumption is likely to be far from the actual condition in shallow heavy trusses with large gusset plates. In such trusses, the joints are more comparable to those in a rigid frame in which the end moments are produced by the continuity that is established through welded or riveted connections. Any error caused by the assumption that the joints of a riveted or welded truss are rigid is partially offset by assuming that the moment of inertia of the members is constant to

the center of the joint—an unnecessary assumption, however, as corrections for the increased moment of inertia within the joint can be approximated by methods explained in the next chapter.

The effect of secondary stresses on the ultimate strength of triangular trusses has never been definitely determined. This subject is discussed in several references at the end of the chapter, throughout which various conflicting opinions will be found. Undoubtedly, the effect and therefore the importance of the secondary stresses will depend on such factors as the manner of loading, type of connections, material used, and the proportions of the members.

6.15 Analytical Procedure

The following procedure is suggested for calculating the end moments and the corrections to the axial stresses that are caused by using rigid joints in triangular trusses.

1. Calculate the axial stresses on the assumption that the ends of the members are pin-connected. The truss is usually loaded with full live load and dead load although certain end moments may be a maximum under partial live load. If such variations are important, calculations can be made for a load at each panel point, although ordinarily such accuracy will not be necessary. The moments due to the dead load may be affected considerably by the method of erection.
2. Draw a Williot displacement diagram, and scale the relative transverse displacement Δ for each member. The rotation of the truss as a rigid body (Mohr rotation diagram) does not change its configuration and consequently does not affect the end moments.
3. Calculate the fixed-end moments $6EI\ \Delta/L^2$ for each member and the distribution factors at each joint.
4. Distribute the fixed-end moments, starting at the joints with the greatest unbalance. Always carry over the correction to the next joint before balancing at that joint, since this procedure increases the rate of convergence. In applying the moment-distribution method, it is, of course, assumed that auxiliary forces are applied at each joint to prevent any additional translation because of flexure in the members.
5. After the end moments have been determined in item 4, the shears in each member and the auxiliary forces at each joint should be calculated. If these auxiliary forces are large, they should be removed by applying equal and opposite forces and the corresponding axial stresses should be determined.

6. Another Williot diagram can then be drawn for the corrections to the axial stresses that are obtained in item 5, and the corresponding end moments, shears, and axial stresses can be obtained as before. The final magnitudes of the end moments, shears, and axial stresses are therefore obtained as a converging series which can be carried to any desired degree of accuracy.

Example 6.10 The application of this method will be illustrated by calculating the secondary stresses for the truss in Example 3.1, Fig. 3.5. The Williot diagram for this truss, shown in Fig. 3.6, requires no additional

Table 6.3

($E = 29 \times 10^6$ lb per square inch)

Member	Length, in.	$K = \frac{I}{L}$ in.^{3}	$\frac{6EI}{L^2} = \frac{174K}{L}$ (10^6)	Δ in.	Fixed-End Moment (10^6 in.-lb)
ab	144	17	20.55	0	0
ae	240	8	5.80	0	0
bc	144	17	20.55	+0.290	+5.95
be	192	8	7.24	+0.022	+0.16
cd	192	12	10.87	+0.008	+0.09
ce	240	6	4.35	+0.210	+0.91
de	144	10	12.10	+0.339	+4.10
df	192	12	10.87	+0.643	+6.97
ef	240	4	2.90	+0.720	+2.09
eg	192	6	5.43	+0.578	+3.14
fg	144	5	6.05	+0.477	+2.88
fh	240	18	13.05	+1.022	+13.35
gh	192	6	5.43	+0.888	+4.82

explanation. The relative displacements and the I/L values as well as the fixed-end moments are tabulated in Table 6.3. The signs of the fixed-end moments were established directly by inspection of the displacement diagram. Distribution of the fixed-end moments is shown in Fig. 6.22*a*, in which the joints were balanced in the following order: *h*, *f*, *g*, *d*, *e*, *c*, *b*, *a*. The second and third cycles were made in the same order. It should be noted that the correction was carried over before the next joint was balanced, and by this procedure sufficient accuracy was obtained at the end of three cycles. The end moments and shears are large, for the truss is short and heavy.

The shear and axial force in each member were calculated from the

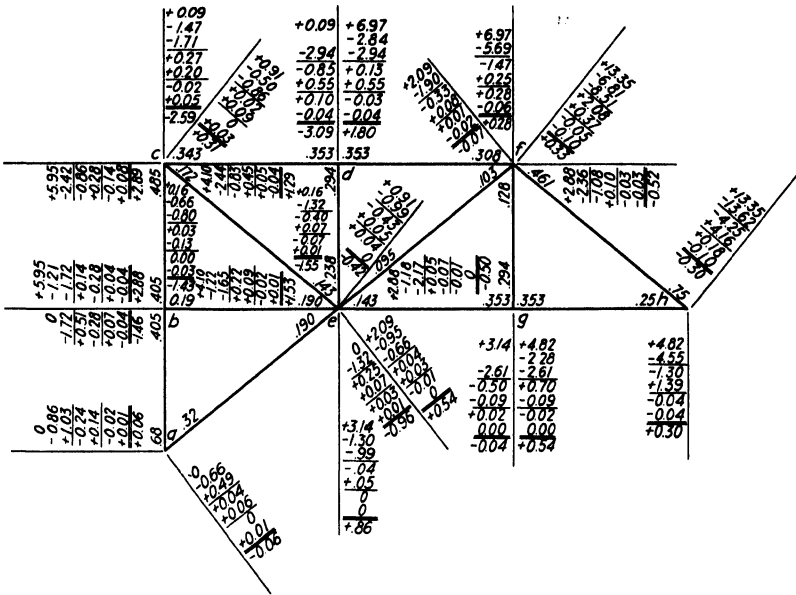



Fig. 6.22a **First approximation for end moments in a triangular truss.**

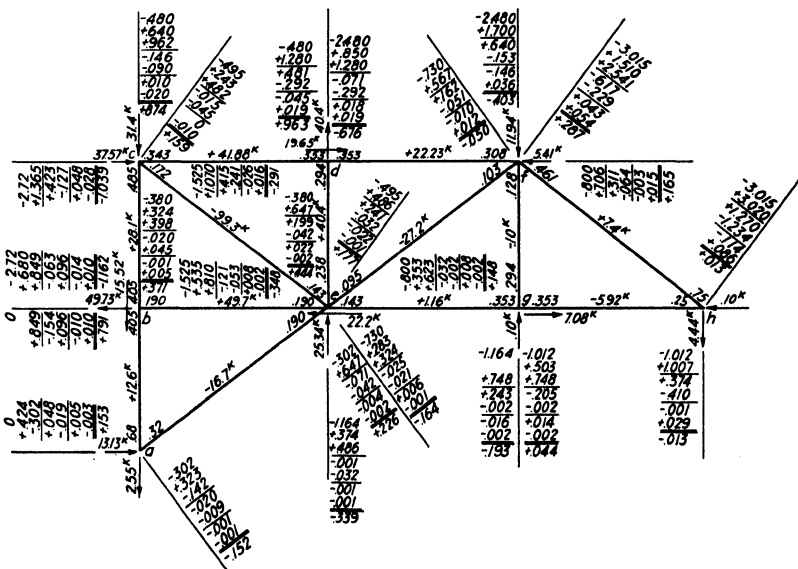



Fig. 6.22b **Second approximation for end moments in a triangular truss.**

Fig. 6.22c Summary of trusses in a triangular truss from successive approximations.

first set of end moments, and by a summation of the horizontal and vertical components of these stresses at each joint the auxiliary forces recorded in Fig. 6.22*b* were obtained. The axial stress is also recorded on the diagram. Another Williot diagram (not shown) was drawn from the change in length of the members due to this axial stress, and another set of fixed-end moments was calculated. These fixed-end moments were distributed, and another set of axial stresses was calculated. The two sets of end moments and the three sets of axial stresses are recorded in Fig. 6.22*c*.

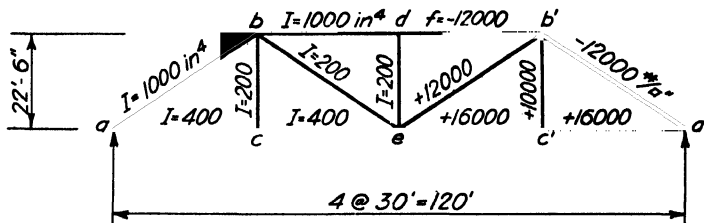
Problems

6.14 (*a*) Calculate the end moments and shears in the members of the truss shown. Use center-to-center distances and constant E.

Problem 6.14

(*b*) Determine the auxiliary forces necessary to restrain the joints against further motion.

(*c*) Record the change in the axial forces if the auxiliary forces are removed.

6.15 Compute the end moments in the Pratt truss shown. The moments of inertia and axial stress are recorded on the diagram of the truss. (*Suggestion:* Take advantage of symmetry by setting θ_d and θ_e equal to zero.)

Problem 6.15

References

1 O. Mohr, "Die Berechnung der Nebenspannungen in Fachwerken mit steifen Knotenverbindungen," *Der Eisenbau*, 1912.

2 C. V. Von Abo, "Secondary Stresses in Bridges," *Trans. Am. Soc. C. E.*, Vol. 89 (1926).

3 John E. Goldberg, "Wind Stresses by Slope Deflection and Converging Approximations," *Trans. Am. Soc. C. E.*, Vol. 99 (1934).

4 L. E. Grinter, "Wind Stress Analysis Simplified," *Trans. Am. Soc. C. E.*, Vol. 99 (1934).

5 F. P. Witmer, "Wind Stress Analysis by the K-Percentage Method," *Trans. Am. Soc. C. E.*, Vol. 107 (1942).

6 J. I. Parcel and E. B. Murer, "Effect of Secondary Stresses upon Ultimate Strength," *Trans. Am. Soc. C. E.*, Vol. 101 (1936).

7 A. Vierendeel, *Cours de stabilite des construction*, Vol. 4, Louvain, 1920.

8 K. Čališev, "Solution of Vierendeel Systems by Successive Approximations," *Diss. Zagrebu, Yugoslavia*, 1921.

9 Dana Young, "Analysis of Vierendeel Trusses," *Trans. Am. Soc. C. E.*, Vol. 102 (1937).

10 David M. Wilson, "Analysis of Rigid Frames by Superposition," *Proceedings, Am. Soc. C. E.*, February 1944.

11 R. Fleming, *Wind Stresses in Buildings*, John Wiley and Sons.

12 J. R. Benjamin, *Statically Indeterminate Structures*, Chapter VI, McGraw-Hill Book Co.

13 R. W. Clough, "Dynamic Effects of Earthquakes," *J. Struct. Div., Am. Soc. C. E.*, April 1960.

14 I. A. El Demirdash, "Statics of the Vierendeel Girder," *Intern. Assoc. for Bridge and Struct. Eng.*, Vol. 12 (1952).

7

Continuous Girders and Frames with Variable Moment of Inertia

7.1 Review of Slope Deflection Relations in Beams

In Chapter 4 the algebraic expressions that relate the end couples M_{ab} and M_{ba} to the corresponding end rotations θ_a and θ_b, the relative transverse displacement Δ_{ab} between the ends, and the applied forces were developed for straight prismatic members. The reader should review the derivation and physical meaning of equations 4.8*a*, *b* and 4.11*a*, *b*. In particular it should be noted that the basic equations 4.1*a* and 4.1*b* were obtained from a continuous linear M/EI diagram for each end couple which naturally implies that EI is constant. However, if the cross section of the member varies, then the M/EI diagram may be nonlinear and, consequently, the coefficients $4(EI/L)$, $2(EI/L)$, and $6(EI/L)$ together with the values of M_{Fab} and M_{Fba} will be modified. In this case equations 4.8*a*, *b* can be written in the following form:

$$M_{ab} = \frac{EI_0}{L}\left[C_1\theta_a + C_2\theta_b - (C_1 + C_2)\frac{\Delta_{ab}}{L}\right] + M_{Fab} \qquad (7.1a)$$

$$M_{ba} = \frac{EI_0}{L}\left[C_2\theta_a + C_3\theta_b - (C_2 + C_3)\frac{\Delta_{ab}}{L}\right] + M_{Fba} \qquad (7.1b)$$

It will be shown that the coefficients C_1, C_2, and C_3 can be determined for any known variation in I with respect to any reference section whose moment of inertia is I_0. Usually the smallest value of I is selected for I_0.

Although any variation in I across the member changes the numerical values of the fixed-end moments from those that were obtained for constant EI, the new fixed-end moments will also be designated by M_{Fab} and M_{Fba}. If it is remembered that new values of the fixed-end moments must be calculated for any assigned variation in EI for a given loading condition, no confusion should occur.

7.2 *Calculation of Coefficients* C_1, C_2, C_3

The numerical procedure that is recommended for determining the new coefficients of the slope deflection equations for members with variable EI is practically the same as the one used in Chapter 4 for members with constant EI. Referring to Figs. 7.1*a, b, c, d*, the combined effects of all displacements will give the following relations.

$$\text{Total end rotation at } a = \theta_a = \theta_a' - \theta_a'' + \frac{\Delta}{L} + \alpha_a \tag{7.2a}$$

$$\text{Total end rotation at } b = \theta_b = -\theta_b' + \theta_b'' + \frac{\Delta}{L} + \alpha_b \tag{7.2b}$$

It should be noted that the α values can be either positive or negative. To transform equations 7.2*a, b* into the form of equations 7.1*a, b*, it is necessary to express θ_a' and θ_b' in terms of M_{ab} and θ_a'' and θ_b'' in terms of M_{ba}. This transformation is easily made in the following manner:

Let β_a' and β_b' be the end rotations for M_{ab} equal to unity (Fig. 7.2*a*).
Let β_a'' and β_b'' be the end rotations for M_{ba} equal to unity (Fig. 7.2*b*).
Then

$$\theta_a' = \beta_a' M_{ab} \qquad \theta_b' = \beta_b' M_{ab}$$
$$\theta_a'' = \beta_a'' M_{ba} \qquad \theta_b'' = \beta_b'' M_{ba}$$

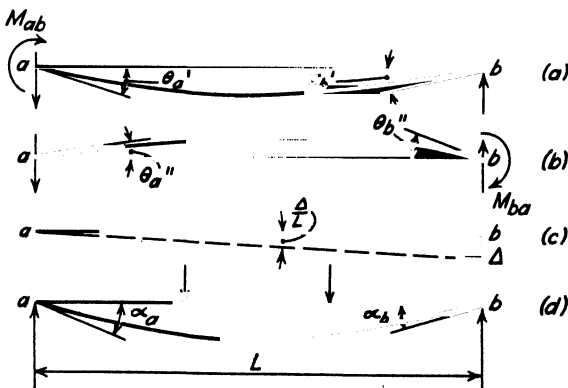



Fig. 7.1

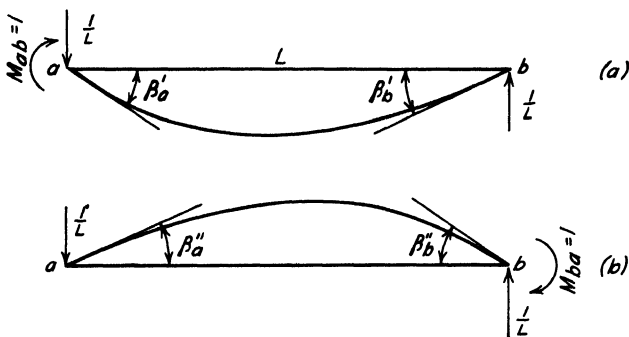



Fig. 7.2

and, as in Chapter 4, if the rotations Δ_{ab}/L and α are first neglected,

$$\theta_a = \theta_a' - \theta_a'' = \beta_a' M_{ab} - \beta_a'' M_{ba} \tag{7.3a}$$

$$\theta_b = -\theta_b' + \theta_b'' = -\beta_b' M_{ab} + \beta_b'' M_{ba} \tag{7.3b}$$

When the reciprocal theorem is applied to the separate force systems shown in Figs. 7.2*a*, *b*, the following relation is obtained:

$$-(1)(\beta_a'') = -(1)(\beta_b')$$

or

$$\beta_a'' = \beta_b'$$

Therefore two of the four β values are always identical regardless of the variation in EI.

When equations 7.3*a* and 7.3*b* are solved for M_{ab} and M_{ba}, the following expressions are obtained:

$$M_{ab} = \frac{\begin{vmatrix} \theta_a & -\beta_a'' \\ \theta_b & \beta_b'' \end{vmatrix}}{\begin{vmatrix} \beta_a' & -\beta_a'' \\ -\beta_b' & \beta_b'' \end{vmatrix}}$$

and

$$M_{ba} = \frac{\begin{vmatrix} \beta_a' & \theta_a \\ -\beta_b' & \theta_b \end{vmatrix}}{\begin{vmatrix} \beta_a' & -\beta_a'' \\ -\beta_b' & \beta_b'' \end{vmatrix}}$$

As it has already been proved that β_a'' is equal to β_b' the preceding expressions reduce to

$$M_{ab} = \frac{\beta_b''}{A}\theta_a + \frac{\beta_b'}{A}\theta_b \tag{7.4a}$$

$$M_{ba} = \frac{\beta_b'}{A}\theta_a + \frac{\beta_a'}{A}\theta_b \tag{7.4b}$$

where

$$A = \beta_a'\beta_b'' - (\beta_b')^2$$

It will be demonstrated that the β angles can be expressed in terms of L/EI_0 so that the denominator A is a function of $(L/EI_0)^2$ and, consequently, each β/A coefficient is equal to a constant times EI_0/L as indicated in equations 7.1a, b. Also, since two β values are equal, then two of the C values must be equal as already indicated. The numerical calculation of the β values and the coefficients C_1, C_2, and C_3 will be illustrated by a numerical example.

Example 7.1 The calculation of the end rotations β and the coefficients C_1, C_2, and C_3 will be made for the beam shown in Fig. 7.3 which has a rectangular cross section with constant width. Therefore

$$I = \left(\frac{d}{1.0}\right)^3 I_0$$

where I_0 is the moment of inertia at end a. Thus at point 4,

$$I = \left(\frac{1.4}{1.0}\right)^3 I_0 = 2.74 I_0$$

All values of d and I are tabulated on the sketch.

To calculate any end rotation for a beam with zero deflection at either

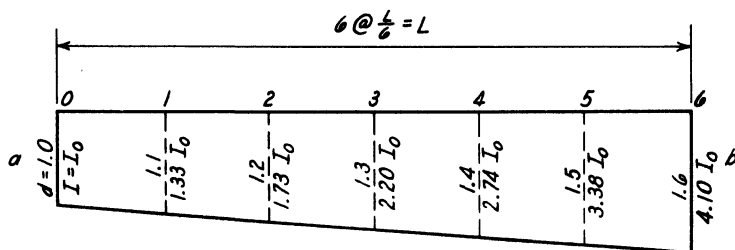



Fig. 7.3

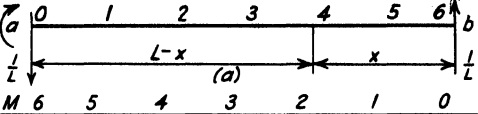
end, such as β_a' and β_b' in Fig. 7.2*a*, it is necessary to recall that (see equation 3.9)

$$\beta_a' = \frac{\frac{1}{E}\int_0^L x\frac{M}{I}\,dx}{L} \tag{7.5a}$$

and

$$\beta_b' = \frac{\frac{1}{E}\int_0^L (L-x)\frac{M}{I}\,dx}{L} \tag{7.5b}$$

Various methods are available for determining the value of the integrals in equations 7.5*a*, *b* and probably a designer should use the procedure that is most familiar to him. The following method which utilizes Simpson's rule to approximate the value of the integrals $\int_0^L x\frac{M}{I}\,dx$ and $\int_0^L (L-x)(M/I)\,dx$ is both accurate and convenient. Usually no more than six sections are needed to give sufficient accuracy.

The necessary data for determining β_a' and β_b' in Fig. 7.4*a* are recorded on the diagram. These are: (1) values of M in terms of a common multiplier $\frac{1}{6}$, (2) values of I in terms of I_0, (3) values of M/I in terms of $1/6I_0$, (4) values of x in terms of $L/6$, and (5) values of $x(M/I)$ in terms of $L/36I_0$. The integral of the $x(M/I)dx$ curve from 0 to L can be accurately obtained from Simpson's rule as follows:

$$\int_0^L x\frac{M}{I}\,dx$$

$$= \left(\frac{L}{6}\right)\left(\frac{1}{3}\right)[36.0 + 4(18.8 + 4.08 + 0.3) + (2)(9.24 + 1.46) + 0]\frac{L}{36I_0}$$

$$= \frac{0.2317L^2}{I_0}$$

Therefore

$$\beta_a' = \frac{1/E(0.2317L^2/I_0)}{L} = \frac{0.2317L}{EI_0}$$

The corresponding values for determining β_b' are shown in Fig. 7.4*b*.

$$\int_0^L (L-x)\frac{M}{I}\,dx$$

$$= \left(\frac{L}{6}\right)\left(\frac{1}{3}\right)[0 + 4(3.76 + 4.08 + 1.50) + 2(4.62 + 2.92) + 0]\frac{L}{36I_0}$$

$$= \frac{0.0809L^2}{I_0}$$

$M_{ab}=1$ a 0 1 2 3 4 5 6 b

$\frac{1}{L}$ $L-x$ x $\frac{1}{L}$

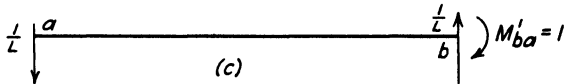
(a)

M	6	5	4	3	2	1	0	Multiplier $\frac{1}{6}$
I	1.0	1.33	1.73	2.20	2.74	3.38	4.10	Multiplier I_0
$\frac{M}{I}$	6.0	3.76	2.31	1.36	0.73	0.30	0	Multiplier $\frac{1}{6I_0}$
X	6	5	4	3	2	1	0	Multiplier $\frac{L}{6}$
$x\frac{M}{I}$	36.0	18.80	9.24	4.08	1.46	0.30	0	Multiplier $\frac{L}{36I_0}$

(b)

$\frac{M}{I}$	6.0	3.76	2.31	1.36	0.73	0.30	0	Multiplier $\frac{1}{6I_0}$
$L-x$	0	1	2	3	4	5	6	Multiplier $\frac{L}{6}$
$(L-x)\frac{M}{I}$	0	3.76	4.62	4.08	2.92	1.50	0	Multiplier $\frac{L}{36I_0}$

$\frac{1}{L}$ a $\frac{1}{L}$ b $M'_{ba}=1$

(c)

M	0	1	2	3	4	5	6	Multiplier $\frac{1}{6}$
$\frac{M}{I}$	0	0.75	1.16	1.36	1.46	1.48	1.46	Multiplier $\frac{1}{6I_0}$
$x\frac{M}{I}$	0	3.75	4.64	4.08	2.92	1.48	0	Multiplier $\frac{L}{36I_0}$

(d)

$L-x$	0	1	2	3	4	5	6	Multiplier $\frac{L}{6}$
$\frac{M}{I}$	0	0.75	1.16	1.36	1.46	1.48	1.46	Multiplier $\frac{1}{6I_0}$
$(L-x)\frac{M}{I}$	0	0.75	2.32	4.08	5.84	7.40	8.76	Multiplier $\frac{L}{36I_0}$

Fig. 7.4

Therefore

$$\beta_b' = \frac{1/E(0.0809L^2/I_0)}{L} = \frac{0.0809L}{EI_0}$$

Similar computations can be made for β_a'' and β_b'' by means of the values recorded in Figs. 7.4*c*, *d*. From these data the following absolute values are obtained:

$$\beta_a'' = \frac{(1/E)(L/6)(1/3)[0 + 4(3.75 + 4.08 + 1.48) + 2(4.64 + 2.92) + 0]L/36I_0}{L} = \frac{0.0808L}{EI_0}$$

$$\beta_b'' = \frac{(1/E)(L/6)(1/3)[0 + 4(0.75 + 4.08 + 7.40) + (2)(2.32 + 5.84) + 8.76]L/36I_0}{L} = 0.1142\frac{L}{EI_0}$$

A partial check on the accuracy of the computations is provided by comparing the numerical values of β_b' and β_a'' which should be identical.

Using these β values in equations 7.4*a, b*, we obtain

$$A = [(0.2317)(0.1142) - (0.0808)^2]\left(\frac{L}{EI_0}\right)^2$$

or

$$A = 0.01993\left(\frac{L}{EI_0}\right)^2$$

$$C_1K_0 = \frac{\beta_b''}{A} = \frac{0.1142}{0.01993}\left(\frac{EI_0}{L}\right) = 5.73\frac{EI_0}{L}$$

$$C_2K_0 = \frac{\beta_b'}{A} = \frac{0.0808}{0.01993}\left(\frac{EI_0}{L}\right) = 4.05\frac{EI_0}{L}$$

$$C_3K_0 = \frac{\beta_a'}{A} = \frac{0.2317}{0.01993}\left(\frac{EI_0}{L}\right) = 11.63\frac{EI_0}{L}$$

7.3 Calculation of the Fixed-End Moments M_{Fab} and M_{Fba}

As soon as the coefficients C_1, C_2, and C_3 are known, the fixed-end moments can be determined in the manner previously discussed in Article 4.4. This procedure is based on the principle that the end couples that will rotate the end sections back through the angles α_a and α_b produced by the applied load acting on a simply supported beam are identical with the fixed-end moments. Therefore M_{Fab} and M_{Fba} are equal to

$$M_{Fab} = \frac{EI_0}{L}[C_1(-\alpha_a) + C_2(-\alpha_b)] \quad (7.6a)$$

$$M_{Fba} = \frac{EI_0}{L}[C_2(-\alpha_a) + C_3(-\alpha_b)] \quad (7.6b)$$

The numerical values of α_a and α_b are conveniently obtained from the M/EI diagram in the same manner as the β values in Example 7.1. This procedure will be illustrated by the following example.

Example 7.2 The fixed-end couples necessary to prevent rotation of the end sections of the beam in Example 7.1 by a uniform load w over the entire span L will be calculated by equations 7.6*a, b*. The bending moment M, in terms of wL^2 for the simply supported beam, as well as the values I, M/I, x, $x(M/I)$, $L - x$, $(L - x)\,M/I$, are recorded in

w lb/ft

a, b; points 0, 1, 2, 3, 4, 5, 6; $L-x$, x

	0	1	2	3	4	5	6	Multiplier
M	0	.0695	.1111	.1250	.1111	.0695	0	wL^2
I	1.0	1.33	1.73	2.20	2.74	3.38	4.10	I_0
$\frac{M}{I}$	0	.0522	.0643	.0568	.0406	.0206	0	$\frac{wL^2}{I_0}$
x	6	5	4	3	2	1	0	$\frac{L}{6}$
$x\frac{M}{I}$	0	.2610	.2572	.1704	.0812	.0206	0	$\frac{wL^3}{6I_0}$
$L-x$	0	1	2	3	4	5	6	$\frac{L}{6}$
$(L-x)\frac{M}{I}$	0	.0522	.1286	.1704	.1624	.1030	0	$\frac{wL^3}{6I_0}$

Fig. 7.5

Fig. 7.5. Again using Simpson's rule we obtain

$$\alpha_a = \frac{(1/E)(L/6)(1/3)[0 + 4(0.2610 + 0.1704 + 0.0206) + 2(0.2572 + 0.0812) + 0]wL^3/6I_0}{L}$$

or

$$\alpha_a = 0.0230\frac{wL^3}{EI_0}$$

$$\alpha_b = -\frac{(1/E)(L/6)(1/3)[0 + 4(0.0522 + 0.1704 + 0.1030) + (2)(0.1286 + 0.1624) + 0]wL^3/6I_0}{L}$$

or

$$\alpha_b = -0.01745\frac{wL^3}{EI_0}$$

Using the coefficients that were determined in Example 7.1, we obtain

$$M_{Fab} = \left(\frac{EI_0}{L}\right)\left(\frac{wL^3}{EI_0}\right)[(5.73)(-0.0230) + (4.05)(0.01745)]$$

or

$$M_{Fab} = -0.0611wL^2$$

$$M_{Fba} = \left(\frac{EI_0}{L}\right)\left(\frac{wL^3}{EI_0}\right)[(4.05)(-0.0230) + (11.63)(0.01745)]$$

or

$$M_{Fba} = 0.1098wL^2$$

In these calculations the values of α_a and α_b must be used with their proper signs as well as magnitudes.

7.4 Members with One End Hinged

If one end of the member is hinged, for example, at a, the moment M_{ab} is zero, and the angle θ_a can be removed from the expression for M_{ba}. Thus, if

$$M_{ab} = K_0\left[C_1\theta_a + C_2\theta_b - (C_1 + C_2)\frac{\Delta_{ab}}{L}\right] + M_{Fab} = 0$$

where $K_0 = EI_0/L$, then

$$\theta_a = -\frac{M_{Fab}}{C_1K_0} - \frac{C_2}{C_1}\theta_b + \frac{C_1 + C_2}{C_1}\frac{\Delta_{ab}}{L}$$

If this value of θ_a is substituted in equation 7.1b, the simplified form of the equation is

$$M_{ba} = K_0\left[\left(C_3 - \frac{C_2^2}{C_1}\right)\left(\theta_b - \frac{\Delta_{ab}}{L}\right)\right] + M_{Fba} - \frac{C_2}{C_1}M_{Fab} \qquad (7.7a)$$

or better

$$M_{ba} = C_b'K_0\left(\theta_b - \frac{\Delta_{ab}}{L}\right) + M'_{Fba}$$

If the beam is hinged at b, the simplified form of the equation for M_{ab} is

$$M_{ab} = K_0\left[\left(C_1 - \frac{C_2^2}{C_3}\right)\left(\theta_a - \frac{\Delta_{ab}}{L}\right)\right] + M_{Fab} - \frac{C_2}{C_3}M_{Fba} \qquad (7.7b)$$

or

$$M_{ab} = C_a'K_0\left(\theta_a - \frac{\Delta_{ab}}{L}\right) + M'_{Fab}$$

Equations 7.7a, b are convenient when it is desirable to change from restrained ends to hinged ends. However, when starting from a hinged condition at one end, say end b, then it is unnecessary to calculate the coefficients C_1, C_2, and C_3 since

$$\theta_a = \beta_a'M_{ab}$$

and therefore

$$M_{ab} = \frac{1}{\beta_a'}\theta_a = C_a'K_0\theta_a \qquad (7.7c)$$

where

$$C_a'K_0 = \frac{1}{\beta_a'} = \left(C_1 - \frac{C_2^2}{C_3}\right)K_0$$

Similarly, the fixed-end moment at a

$$M'_{Fab} = M_{Fab} - \frac{C_2}{C_3} M_{Fba}$$

is equal to

$$M'_{Fab} = C_a' K_0(-\alpha_a) \tag{7.7d}$$

Example 7.3. If the beam in Examples 7.1 and 7.2 is hinged at end b, then the slope-deflection equation for M_{ab} can be written in the form

$$M_{ab} = C_a' \frac{EI_0}{L}\left(\theta_a - \frac{\Delta_{ab}}{L}\right) + M'_{Fab}$$

where

$$C_a' \frac{EI_0}{L} = \frac{1}{\beta_a'} = \frac{1}{0.2317L/EI_0} = 4.32 \frac{EI_0}{L}$$

$$M'_{Fab} = -4.32 \frac{EI_0}{L} \alpha_a = \left(4.32 \frac{EI_0}{L}\right)\left(-0.023 \frac{wL^3}{EI_0}\right)$$

or

$$M'_{Fab} = -0.0993wL^2$$

These values may also be obtained by substituting the results obtained in Examples 7.1 and 7.2 for C_1, C_2, C_3, M_{Fab}, and M_{Fba} in equation 7.7b.

$$M_{ab} = \frac{EI_0}{L}\left(5.73 - \frac{4.05^2}{11.63}\right)\left(\theta_a - \frac{\Delta_{ab}}{L}\right) + \left(-0.0611 - \frac{4.05}{11.63}\,0.1094\right)wL^2$$

or

$$M_{ab} = 4.32 \frac{EI_0}{L}\left(\theta_a - \frac{\Delta_{ab}}{L}\right) - 0.0992wL^2$$

7.5 *Charts for Coefficients C_1, C_2, and C_3*

The procedure described for computing the values of the coefficients C_1, C_2, and C_3 in terms of some reference section whose moment of inertia is I_0 is by no means difficult, but it is usually time-consuming if many members are involved. For this reason tables and charts giving the values of the coefficients and fixed-end moments for particular types of members are often constructed (see Refs. 1 and 3). In general, these charts are for members having rectangular cross sections with constant width and with depths that vary according to some definite shape, such as (1) beams with straight or parabolic haunches at one or both ends, and (2) beams with infinite cross section at one or both ends. However, recent technical literature contains numerous charts from which coefficients may be obtained for beams with I-shaped cross sections.

Members with other types of cross section can frequently be solved by converting them into members with equivalent rectangular sections, that is, rectangular cross sections with the same variation in I. This can be done by calculating the value of I for the actual member at several sections and then determining the equivalent depth d for the substitute member from the relation that d/d_0 is equal to $\sqrt[3]{I/I_0}$, where d_0 and I_0 refer to the reference section. Approximate values of the coefficients can then be obtained from the charts for rectangular sections. The use of the diagrams in the Appendix will be illustrated later by numerical examples.

7.6 *Effect of Construction Details on C_1, C_2, and C_3*

The preceding discussion of the determination of the coefficients C_1, C_2, and C_3 does not consider the practical problem of how the deformation of material within the joint, or, rather, inside a zone that is more or less common to several intersecting members, affects their values. A more comprehensive discussion of this subject as related to practical design will be presented later, but at the present time the following assumptions seem warranted by the limited experimental results that are available.

Rectangular Cross Sections. When structural members with rectangular cross section intersect with sharp corners, Fig. 7.6*a*, the effect of the deformation of the member within the joint can be obtained with reasonable accuracy by assuming a constant cross section to the center lines of the members. The omission of the deformations because of shear and stress concentration more than compensates for any increase in depth within the joint. For the calculation of bending moments, the assumption of constant cross section is therefore sufficiently accurate. For tapered members with sharp corners, Fig. 7.6*b*, it is sufficiently accurate to continue the inclined edge to the center line of the members. For members with curved haunch, Fig. 7.7, it is recommended that the material within the joint, shown by the shaded area, be assumed as having no deformation or that I equals infinity. In the curved portion the value of I is computed

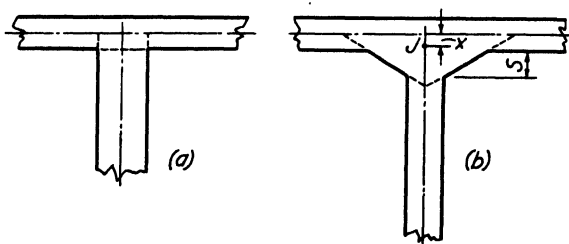



Fig. 7.6 Effective cross section within the joint.

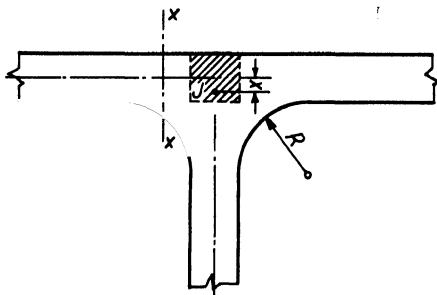



Fig. 7.7 Joints with curved fillets.

for the transverse section x-x by assuming that the full cross section is effective, although the stress distribution is by no means linear. This variation from a linear stress distribution is particularly important when flange sections are used.

The theoretical center of the joint is at some point J slightly below the intersection of the center lines as shown in Figs. 7.6 and 7.7. Tests made by the author on celluloid models showed that the value of x is not more than $S/8$ for straight 45° haunches or $R/8$ for circular ones. The intersection of the bottom of the straight portion with the center line of the column is recommended as a working point for relatively deep haunches.

I-Shaped Sections. Both theoretical and experimental data prove that in members which vary in depth and which are subjected to flexure, the distribution of normal stress is a nonlinear one (see Ref. 8). In a member which has a rectangular cross section of constantly varying depth and which is subjected to pure flexure, the normal and shearing stresses are distributed across the section as indicated in Fig. 7.8*a*. An exact solution of this problem shows that the normal stress f_x on the outer fibers is practically equal to $f' \cos^3 \alpha$, in which f' is the stress computed by the usual flexure formula on the basis of a linear distribution of stress. If flanges are added, the distribution across the web is not materially altered, and the average stress in the flange will not be more and is probably less than at the edge of the web. Such a distribution would give, on a section normal to the axis of an I member, such as section 1-1, Fig. 7.8*b*, a total flange component N parallel to the axis equal to $f_x(A_f/\cos \alpha)$, where α is the angle of inclination of the flange to the axis. Since f_x is equal to $f' \cos^3 \alpha$, this total flange component N equals $A_f f' \cos^2 \alpha$, and therefore if a straight-line distribution of stress is assumed, an equivalent flange area of $A_f \cos^2 \alpha$ should be used to determine the effective cross section. The web area can be used without reduction.

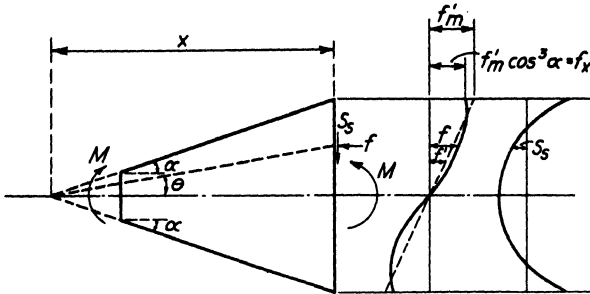



Fig. 7.8a Stress distribution in tapered members for pure bending.

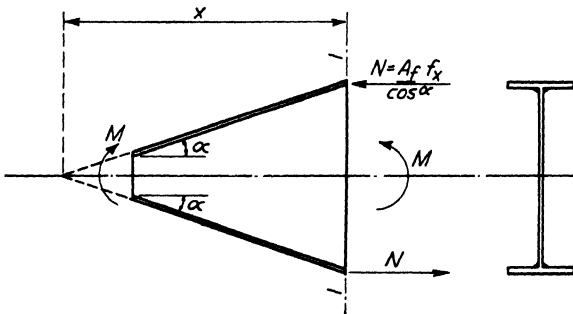



Fig. 7.8b

The principal stress p in the flange parallel to the boundary is

$$p = \frac{f_x}{\cos^2 \alpha} = f' \cos \alpha$$

In computing the coefficients C_1, C_2, and C_3 of the slope deflection equations, the reduced flange area $A_f \cos^2 \alpha$ should be substituted for the actual area. If curved flange sections without radial stiffeners are used, there will be an additional reduction of the effective flange area because of the variation in stress resulting from inward radial bending. This effect can be neglected in computing the coefficients, although it is important in stress and stability calculations.

Although the equivalent flange area $A_f \cos^2 \alpha$ was determined from the actual stress distribution in a straight tapered section under pure bending, experimental data indicate that it is also a good approximation for curved flange sections. The normal stress on sections perpendicular to the axis of the straight portion of the member, which is considered here, should not be confused with the normal stress on radial sections. This latter distribution is entirely different, for it increases toward the boundary. If concentrated loads are applied in the vicinity of curved flanges, shear

and direct compression stresses may alter considerably this distribution for pure bending.

7.7 *Distribution and Carry-Over Factors*

Once the coefficients of the slope deflection equations are known for all members of a frame, the distribution and carry-over factors can be determined in the same manner as described in Article 4.7 for members with constant EI. Thus, as in equation 4.22, the end moments for the members in Fig. 4.13 can be expressed in the following form:

$$\begin{aligned} M_{xa} &= M_{Fxa} + C_{xa}K_{xa}\theta_x \\ M_{xb} &= M_{Fxb} + C_{xb}K_{xb}\theta_x \\ M_{xc} &= M_{Fxc} + C_{xc}K_{xc}\theta_x \\ M_{xd} &= M'_{Fxd} + C_{xd}'K_{xd}\theta_x \end{aligned} \tag{7.8}$$

in which the fixed-end moments and coefficients for members xa, xb, and xc are selected for θ_a, θ_b, and θ_c equal zero, whereas for member xd the values of M'_{Fxd} and C_{xd}' are calculated for M_{dx} equals zero. As already explained in Article 7.4,

$$M'_{Fxd} = M_{Fxd} - \frac{C_2}{C_3} M_{Fdx}$$

and

$$C_{xd}' = C_1 - \frac{C_2^{\,2}}{C_3}$$

or

$$C_{xd}' = C_3 - \frac{C_2^{\,2}}{C_1}$$

Since the summation of all applied moments at a joint must equal zero, the following relation between the rotation θ_x and the fixed-end moments in equation 7.8 must exist.

$$\theta_x \sum CK_0 + \sum M_{Fx} = 0$$

or

$$\theta_x = - \frac{\sum M_{Fx}}{\sum CK_0}$$

Therefore the corrections to the fixed-end moments produced by the rotation θ_x are

$$C_{xa}K_0\theta_x = \frac{C_{xa}K_0}{\sum CK_0}\left(-\sum M_{Fx}\right) = r_{xa}\left(-\sum M_{Fx}\right)$$

$$C_{xb}K_0\theta_x = \frac{C_{xb}K_0}{\sum CK_0}\left(-\sum M_{Fx}\right) = r_{xb}\left(-\sum M_{Fx}\right)$$

or, in general, for the rotation of any joint the change in the end moment is

$$\Delta M_x = r(-\sum M_F)$$

in which the distribution factor r for each member at the joint is

$$r = \frac{C_x K_0}{\sum C_x K_0} \tag{7.9}$$

The value of C_x is always the coefficient of θ_x for any member at joint x for a specified boundary condition at the opposite end.

For any rotation θ_x the fixed-end moment at the opposite fixed end is modified by the quantity $C_2 K_0 \theta_x$ as given in the slope deflection equation. Thus for member ax (Fig. 4.13) for any rotation θ_x

$$M_{ax} = M_{Fax} + C_2 K_{ax} \theta_x = M_{Fax} + \Delta M_{ax}$$

where

$$\Delta M_{ax} = C_2 K_{ax} \theta_x$$

but, since

$$\Delta M_{xa} = C_1 K_{ax} \theta_x$$

then

$$\frac{\Delta M_{ax}}{\Delta M_{xa}} = \frac{C_2 K_{ax} \theta_x}{C_1 K_{ax} \theta_x} = \frac{C_2}{C_1}$$

or

$$\Delta M_{ax} = \frac{C_2}{C_1} \Delta M_{xa} \tag{7.10}$$

In other words, the carry-over factors that are used to determine the change in moments at the ends opposite from the one that is rotated are given by the ratios C_2/C_1 or C_2/C_3. If any confusion occurs as to which of these two ratios to use, the slope deflection equations for the end moments should be written and then one rotation set equal to zero. The ratio of the end moments is then apparent. The application of the moment-distribution method to members with variable cross section will be illustrated by a numerical example.

Example 7.4 The end moments acting on the various members of the frame in Fig. 7.9 will be computed by the moment-distribution method. The coefficients and fixed-end moments for members *ab* and *bc* can be obtained from the diagrams on pages 426–431 in the Appendix, which were prepared by the Portland Cement Association. In these diagrams the coefficients C_1 and C_3 are designated by k, the stiffness coefficient, and the carry-over factors C_2/C_1 and C_2/C_3 by C. The coefficients for the columns *bd* and *ce* will be calculated by the method used in Example 7.1, except that the M/I area will be divided into triangles for calculating the β values.

Fig. 7.9 Analysis of a continuous frame with members having variable I.

Member ab. If the member is first assumed to be restrained at both ends, the coefficient $C_1 = k$ from page 429, diagram 1 ($a = 1.0$, min d/max $d = 0.5$), is 6.9, and the carry-over factor $C = C_2/C_1 = 0.84$ from diagram 2. The value of C_2 is therefore $(0.84)(6.9) = 5.80$. From diagram 3, the fixed-end moment at a is

$$M_{Fab} = -0.053wL^2 = -(0.053)(1)(10)^2 = -5.3 \text{ ft-kips}$$

From page 428, for the haunched end, $k = C_3 = 19.4$ and $C = C_2/C_3 = 0.3$. Therefore $C_2 = (0.3)(19.4) = 5.82$, which is sufficiently close to the value previously given.

The fixed-end moment at b is

$$M_{Fba} = 0.122wL^2 = (0.122)(1)(10)^2 = +12.2 \text{ ft-kips}$$

Since the end a is free to rotate, equation 7.7a should be used, that is,

$$M_{ba} = \left(19.4 - \frac{5.8^2}{6.9}\right)K_0\left(\theta_b + \frac{\Delta}{L}\right) + 12.2 - (0.84)(-5.3)$$

or, since Δ is zero

$$M_{ba} = 14.5K_0\theta_b + 16.65$$

The value of K_0 if ab has a rectangular cross section with a width of one foot is

$$K_0 = \frac{I_0}{L} = \frac{(1.0)^4}{(12)(10)} = 0.00833$$

Member bc. From page 426, diagram 1, for $a = 0.2$, min d/max $d = 0.5$, the coefficients are $k = C_1 = C_3 = 7.8$.
From diagram 2

$$C = \frac{C_2}{C_1} = \frac{C_2}{C_3} = 0.66 \qquad \text{Therefore } C_2 = (0.66)(7.8) = 5.15$$

The fixed-end moments are (diagrams 3 and 4)

$$M_{Fbc} = -[(0.154)(10)(16) + (0.0992)(1)(16)^2] = -50.0 \text{ ft-kips}$$
$$M_{Fcb} = +50.0 \text{ ft-kips}$$

Member bd. The coefficients for member bd will be computed on the assumption that I equals ∞ for a distance of 1.0 ft at the top of the column. For a moment of 12 ft-kips applied at end of d for the column bd, the M/I values are shown in the diagram in Fig. 7.10a. The M/I

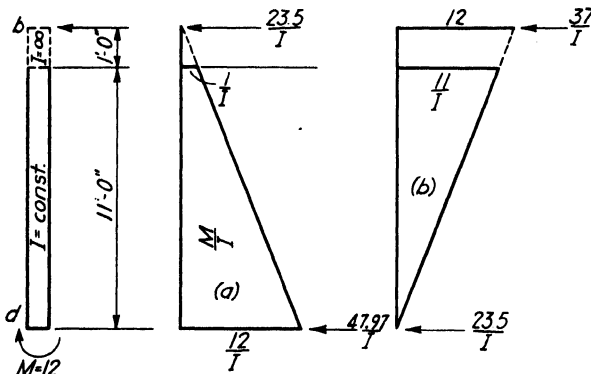



Fig. 7.10

diagram for a moment of 12 ft-kips at b is shown in Fig. 7.10b. The reactions of the conjugate beam when loaded with these diagrams determine the end rotations. For a unit end moment and any span L, the end rotations must be multiplied by $L/144$ or

$$\beta_{d1} = \frac{47.97L}{144EI} = 0.333\frac{L}{EI}$$

$$\beta_{b1} = \beta_{d2} = \frac{23.5L}{144EI} = 0.1632\frac{L}{EI}$$

$$\beta_{b2} = \frac{37.0L}{144EI} = 0.257\frac{L}{EI}$$

$$A = (0.333)(0.257) - (0.163)^2 = 0.0590$$

$$C_1 = \frac{\beta_{b2}}{A} = \frac{0.257}{0.0590} = 4.36$$

$$C_2 = \frac{\beta_{b1}}{A} = \frac{0.1632}{0.0590} = 2.77$$

$$C_3 = \frac{\beta_{d1}}{A} = \frac{0.333}{0.0590} = 5.66$$

Member ce. The coefficients or member ce will be determined in the same manner as for bd, except that $I = \infty$ will be taken for a distance of 0.8 ft instead of 1.0 ft. The values of the coefficients are

$$C_1 = 4.2 \qquad C_2 = 2.6 \qquad C_3 = 5.3$$

Table 7.1 Distribution and Carry-Over Factors

Joint b

Member	C	K_0	CK_0	$r = \frac{CK_0}{\Sigma CK_0}$	Carry-Over
ba	14.5	0.00833	0.121	0.410	0
bc	7.8	$\frac{(1.0)^3}{(12)(16)} = 0.0052$	0.041	0.139	0.66
bd	5.66	$\frac{(1.5)^3}{(12)(12)} = 0.0234$	0.133	0.451	0.49
			$\Sigma CK_0 = 0.295$	$\Sigma r = 1.000$	

Joint c

Member	C	K	CK_0	$r = \frac{CK_0}{\Sigma CK_0}$	Carry-Over
cb	7.8	0.0052	0.041	0.249	0.66
ce	5.3	0.0234	0.124	0.751	0.49
			$\Sigma CK_0 = 0.165$	$\Sigma r = 1.000$	

After the fixed-end moments, distribution factors, and carry-over factors are determined, the solution is made in the same manner as for members with constant cross section. An auxiliary force F is applied to prevent sidesway while the fixed-end moments due to the applied loads are corrected for rotation of the joints. This part of the solution is presented in Fig. 7.9*b*. The value of F must equal the difference between H_d and H_e or 2.64 kips.

To remove the auxiliary force F, the frame will be given some arbitrary horizontal movement of 100 units to the left, for which the fixed-end moments are

$$M_{Fbd} = (C_2 + C_3)K_0\frac{\Delta}{L} = (2.77 + 5.66)(0.0234)\left(\frac{100}{12}\right) = 1.65$$

$$M_{Fdb} = (C_1 + C_2)K_0\frac{\Delta}{L} = (4.36 + 2.77)(0.0234)\left(\frac{100}{12}\right) = 1.39$$

$$M_{Fce} = (5.3 + 2.6)(0.0234)\left(\frac{100}{12}\right) = 1.54$$

$$M_{Fec} = (4.2 + 2.6)(0.0234)\left(\frac{100}{12}\right) = 1.33$$

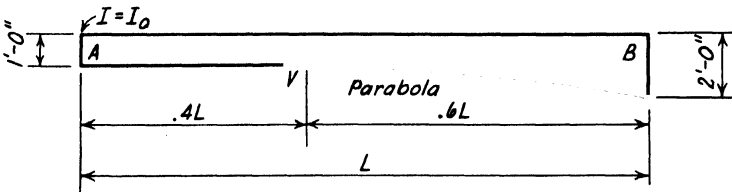
The distribution of these fixed-end moments is shown in Fig. 7.9*c*.

The force F' that is necessary to give these moments is found to be

$$F' = 0.175 + 0.107 = 0.282 \text{ kip}$$

Therefore the force F can be removed by multiplying the moments in Fig. 7.9*c* by 2.64/0.282 and adding the resulting moments to the corresponding values in Fig. 7.9*b*. The final moments are shown in Fig. 7.9*d*. The reader should work this example and similar problems for himself until all the numerical operations are thoroughly understood.

Problems

7.1 (*a*) Determine the values of the coefficients C_1, C_2, C_3 for the beam shown. Assume that the cross section is rectangular with constant width and that the parabolic haunch has a horizontal tangent at point V.

Problem 7.1

(*b*) Calculate the fixed-end moments for a uniform load of w lb per linear foot over the entire span. Assume that both ends are fixed.

(*c*) What is the value of the fixed-end moment at end B if end A is simply supported?

(*d*) What are the numerical values of the carry-over factors?

(*e*) Compare your results with the corresponding values from the P.C.A. diagrams in the Appendix.

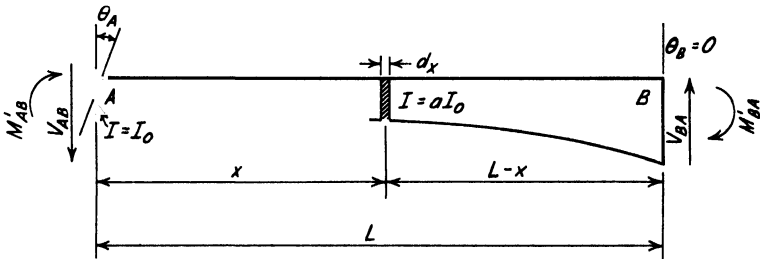
7.2 Determine the vertical displacement at point V for the fixed-end condition in Problem 7.1(*b*) by means of Newmark's method (Article 3.9). Express in terms of wL^4/EI_0.

7.3 (*a*) Prove by Castigliano's theorem that the carry-over factor from A to B when Δ_A and θ_B are zero is given by the expression

$$\text{C.O.}_{AB} = \frac{M'_{BA}}{M'_{AB}} = \frac{\displaystyle\int_0^L \frac{(L-x)x\,dx}{a}}{\displaystyle\int_0^L \frac{x^2}{a}\,dx}$$

where

$$a = \frac{I}{I_0}$$

Discuss how the integration may be replaced by a summation in terms of Δx.

Problem 7.3

(*b*) Repeat part *a* for the value of the stiffness coefficient by showing that

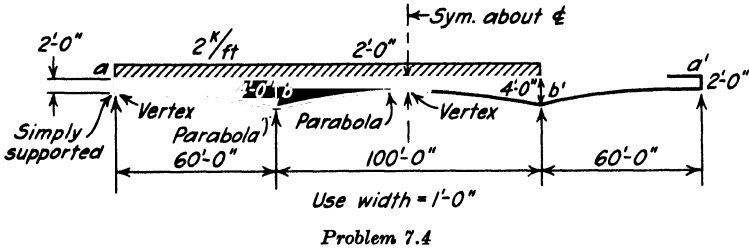
$$M'_{AB} = C_A \frac{EI_0}{L} \theta_A$$

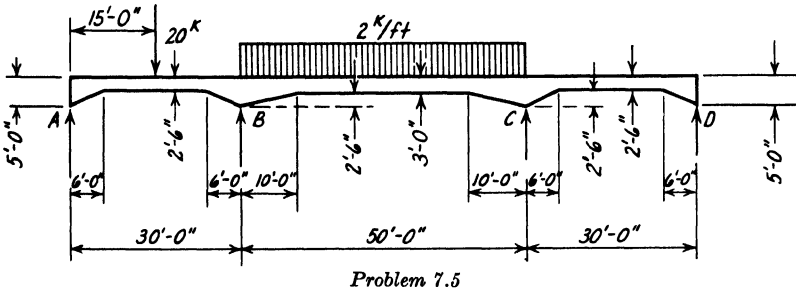
where

$$C_A = C_1 = \frac{L^2}{\displaystyle\int_0^L \frac{(L-x)}{a}\,dx - \text{C.O.}_{AB}\int_0^L \frac{x}{a}\,dx}$$

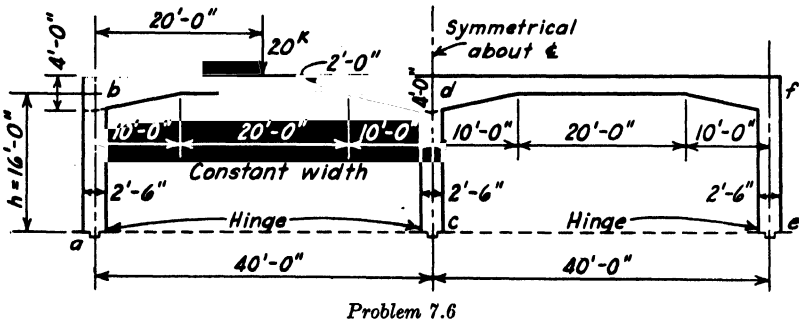
7.4 (*a*) Calculate the end moments for the continuous girder shown by means of the moment-distribution method. Use the coefficients, carry-over factors, and fixed-end moments that are given in the Appendix.

(*b*) Compare the bending-moment diagrams obtained from part *a* with the corresponding diagrams for a girder with constant I.

Problem 7.4

7.5 Determine the bending moments at B and C and the maximum positive moments in spans AB and BC for the girder shown. Use diagrams in the Appendix. Consider width of one foot.

Problem 7.5

7.6 Compute the value of the end moments for all members of the continuous frame shown. Use the P.C.A. diagrams for girder coefficients and fixed-end moments and the following coefficients for the abutments and pier.

$$C_1(\text{top}) = 7.0 \qquad C_2 = 3.5 \qquad C_3(\text{bottom}) = 4.8$$

Problem 7.6

7.7 Prepare a computer program to determine the coefficients C_1, C_2, and C_3 for any beam with variable moment of inertia. Use the Newmark procedure for calculating angle changes. Assume that the ratio I/I_0 is known.

7.8 Extend the computer program in Problem 7.7 to determine the fixed-end moments for a uniform load over the entire span.

7.8 Influence Diagrams for Fixed-End Moments

The method explained in Article 7.3 for calculating fixed-end moments can be followed for determining the ordinates to the influence diagrams for fixed-end moments of beams with either constant or variable cross section. This algebraic solution is somewhat laborious, particularly for beams with variable cross section, and for such members the following solution by the Müller-Breslau principle is recommended. Before this solution can be started, it is necessary that the coefficients C_1, C_2, and C_3 be calculated by the methods already described.

The basis of the Müller-Breslau solution depends upon the relation between elastic curves and influence diagrams. The proof of this method for constructing influence diagrams, which was developed by Müller-Breslau, was discussed in Example 2.2 and will be restated briefly before its application is considered. The force system in Fig. 7.11*a* shows a load P, which is acting at a distance x from one end, and the fixed-end moments M_{Fab} and M_{Fba}, which are to be determined. If an auxiliary force system, consisting of an applied moment M_{ab}', Fig. 7.11*b*, which rotates the tangent at a through an angle θ_a, and an end moment M_{ba}', which prevents any rotation at b, acts on the member, an elastic curve is formed that has a vertical displacement of y at any distance x from the support a. By the reciprocal theorem, which is expressed in algebraic terms by equations 2.9 and 2.10, the following relation is obtained between the forces of Fig. 7.11*a* and the displacements of Fig. 7.11*b*.

$$-M_{Fab}\theta_a + Py = M_{ab}' \cdot 0 + M_{ba}' \cdot 0 = 0$$

or

$$M_{Fab} = P\frac{y}{\theta_a} \tag{7.11}$$

The ordinates to the influence diagram for M_{Fab} of the actual force system in Fig. 7.11*a* are therefore equal to the ratio y/θ_a of the elastic curve produced by the auxiliary forces in Fig. 7.11*b*. The ratio y/θ_a can be obtained by either numerical or graphical methods. A convenient graphical solution for determining the ratio y/θ_a for the auxiliary force system will now be explained.

In terms of the beam coefficients C_1, C_2, and C_3 the end moments in Fig. 7.11*b* have the relation

$$M_{ba}' = \frac{C_2}{C_1} M_{ab}'$$

and the bending-moment diagram is a straight line as shown in Fig. 7.11*c*. The next step consists of drawing the M/I diagram (Fig. 7.11*d*), as was

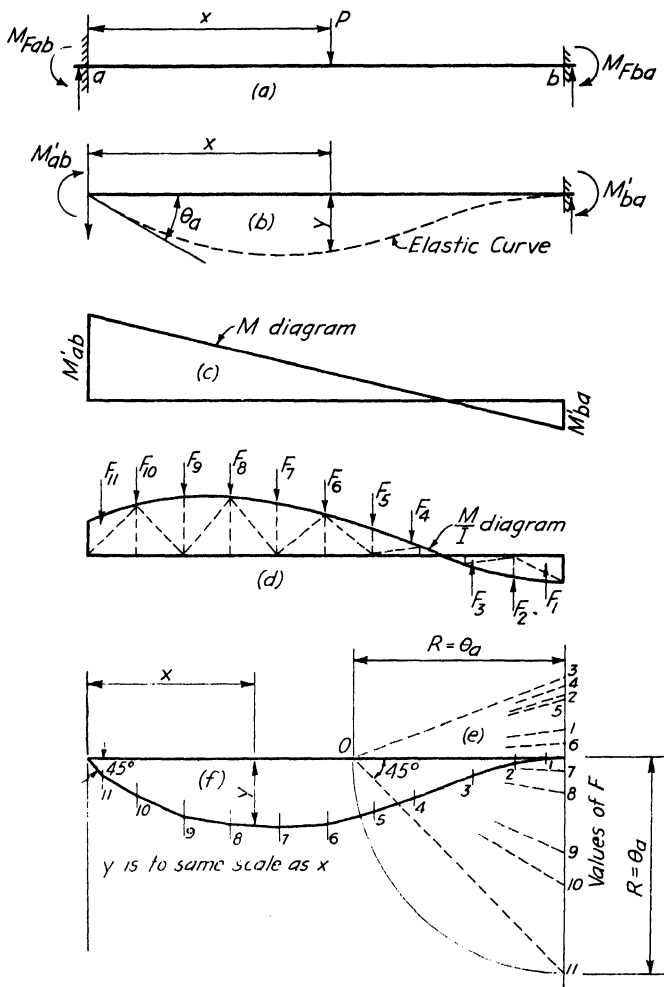



Fig. 7.11 Graphical solution for fixed-end moments in a beam with variable I.

done in the previous calculation for the coefficients, and dividing the diagram into a number of areas. In this illustration triangular areas are used. According to the theorem of area moments these areas represent the change in slope of the elastic curve over a distance equal to the base of the area while, since θ_b is equal to zero, the total area of the diagram is numerically equal to the end rotation θ_a. As the diagram is drawn for E equal to unity, the computed displacements are E times the actual ones; but since we are interested only in the ratio of displacements, the absolute values are not necessary.

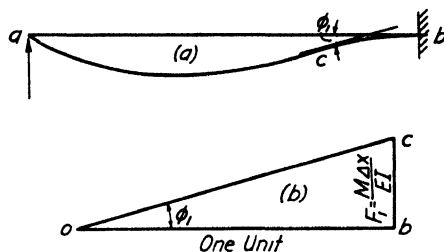



Fig. 7.12 Graphical determination of slope from the area of the M/EI diagram.

If the areas into which the M/I diagram is divided, that is, the $\Delta\phi$ values, are applied as a system of elastic weights F, a force polygon, Fig. 7.11*e*, can be drawn to any convenient scale, beginning with F_1 near the fixed end *b*. The resultant force R is equal to the reaction of the conjugate beam at point *a*, which in turn is equal to the end rotation θ_a. A pole distance equal to θ_a is then measured horizontally from the point at which the force polygon is started to give the pole O. If a funicular polygon (Fig. 7.11*f*), is drawn from the corresponding rays in the force polygon, the ordinates to this diagram are the desired ratio y/θ_a to the same scale as the span of the beam is drawn.

To visualize the physical meaning of the preceding statements it should be kept in mind that all angular rotations are small, and therefore the angle in radians can be represented accurately by its tangent. For this reason, any angle such as ϕ_1 (Fig. 7.12*a*) that the tangent at point *c* makes with the tangent at *b* can be obtained graphically by drawing an angle (Fig. 7.12*b*) whose side *bc* is equal to F_1, the area of the M/EI diagram between points *b* and *c*, and *ob* is unity. Now if the side *ob* is taken equal to θ_a instead of unity, the angle which is still equal to the tangent is

$$\frac{F_1}{\theta_a} = \frac{1}{\theta_a}\phi_1$$

Consequently, all slopes to the elastic curve will be multiplied by $1/\theta_a$ if a pole distance equal to θ_a is used in drawing the funicular polygon, and all ordinates to the elastic curve will be equal to y/θ_a. The elastic curve formed by this funicular polygon is therefore the influence diagram for M_{Fab} to the same scale as the beam is drawn. However it is apparent that if a pole distance of θ_a/n is used instead of θ_a, then the ordinates to the elastic curve are $n(y/\theta_a)$, and the scale ratio is therefore n times the scale of the span. The ordinates of this elastic curve are independent of the scale to which the force polygon for the M/I areas is drawn. As already stated, the ordinates are also independent of the value of E provided it is constant.

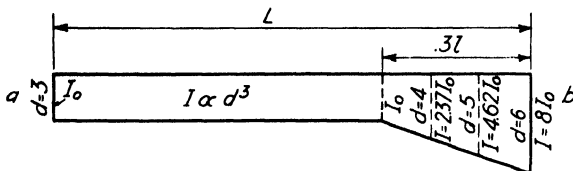



Fig. 7.13

Example 7.5 An influence diagram for the fixed-end moment M_{Fab} of the beam in Fig. 7.13 will be constructed by the graphical method. The coefficients for this beam which can be determined in the usual manner are

$$C_1 = 4.68 \qquad C_2 = 3.68 \qquad C_3 = 8.13$$

A moment M_{ab}' equal to 10 is applied at a (Fig. 7.14a) and the moment M_{ba}' that is required to prevent rotation at b is

$$M_{ba}' = \frac{3.68}{4.68} 10 = 7.86$$

The M/I diagram for these end moments is shown in Fig. 7.14a, together with the areas F, into which the diagram is divided. It should be noted that the use of the triangular areas F into which the M/I diagram is divided results in a small error in the calculations of the bending moments in the conjugate beam at the points where the F forces are applied. The bending moments in the conjugate beam and therefore the displacements in the actual beam are slightly high, for the moment of a small part of the M/I diagram is neglected, but this effect is negligible. The end rotation θ_a is represented by the concentrated force R_a which is equal to the algebraic sum of the areas, 21.34. There is no reaction at the end b of the conjugate beam, for the rotation at that end is zero. The areas are laid off to scale in the force polygon of Fig. 7.14b, and a pole distance equal to R_a or θ_a is drawn. The funicular polygon (Fig. 7.14c) that is drawn from the force polygon is the influence diagram for M_{Fab} to the same scale as was used for the span L. In this problem the span L was taken equal to 10 ft, and a scale of 1 in. to 1 ft was adopted in drawing the diagram. Therefore 1 in. of vertical distance in the influence diagram is equal to 1 ft-lb for a 10-ft span. For any span L and any load P the scale of the influence diagram will be 1 in. equals $(1/10)PL$ ft-lb.

Although the scale chosen in this problem is not large, the accuracy is equal to that obtained in an algebraic solution. The ordinate at the center of the span measured 0.89 in., and therefore the fixed-end moment for a concentrated load P at that point is $0.089PL$, which checks the value obtained algebraically.

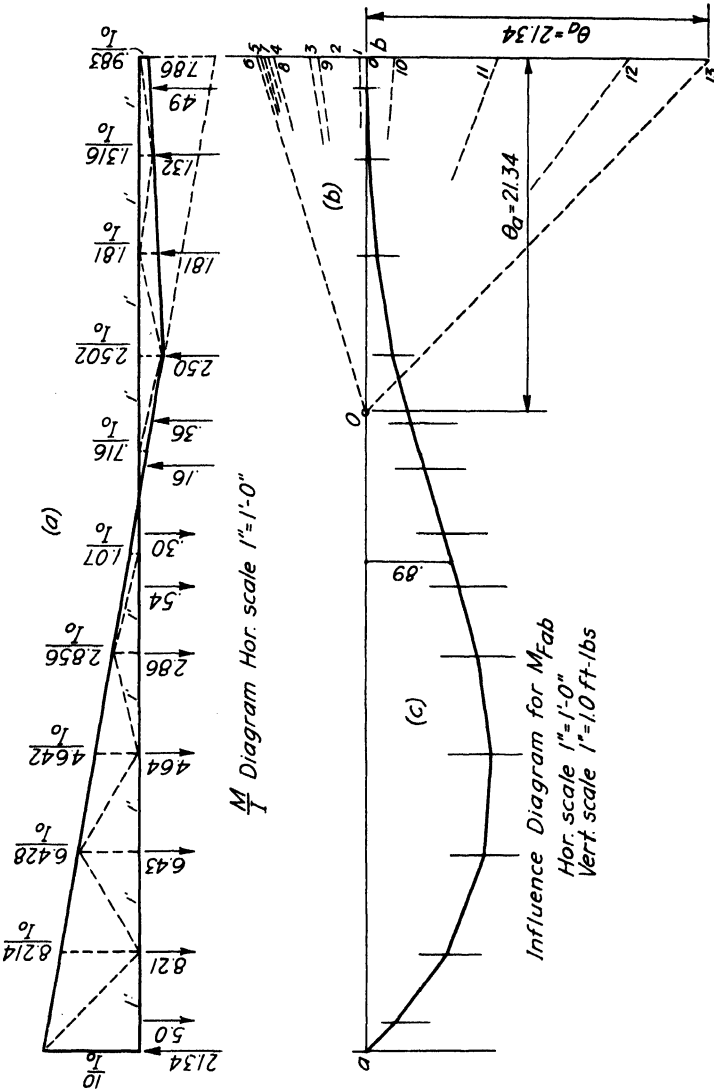



Fig. 7.14 Graphical solution for the influence diagram for M_{Fab} when end b is fixed.

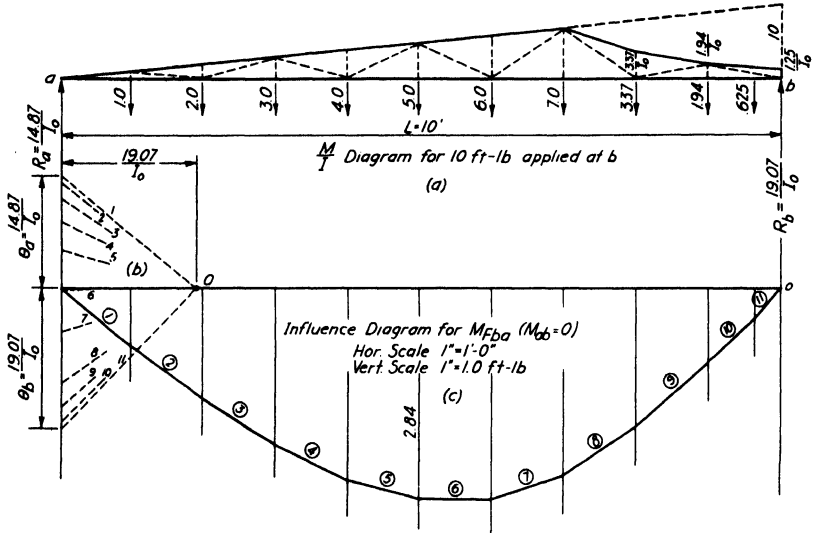



Fig. 7.15 Graphical solution for the influence diagram for M_{Fba} when end a is hinged.

The influence diagram for M'_{Fba} when the end a is simply supported is shown in Fig. 7.15. Here a moment of M_{ba}' equal to 10 units is applied at b and the moment M_{ab}' at a is zero. The conjugate beam that is loaded with the M/I diagram is shown in Fig. 7.15*a*. The reactions R_a and R_b of the conjugate beam equal the end rotations θ_a and θ_b. Therefore, if a force polygon (Fig. 7.15*b*) is drawn for all forces acting on the conjugate beam and a pole distance equal to R_b or θ_b is used, the resulting funicular polygon (Fig. 7.15*c*) is the influence diagram for the fixed-end moment at b when end a is free to rotate. These ordinates are again to the same scale as that used for drawing the span of the beam. The ordinate at the center of the span is equal to $0.284PL$ as compared to the value of $0.287PL$ obtained by an algebraic solution.

The values of y/θ_a, that is, the ordinates to the influence diagram, can also be obtained algebraically if desired since the displacement y is numerically equal to the bending moment in the conjugate beam. Thus the ordinate at the center of the span in Fig. 7.14 is equal to

$$\frac{y}{\theta_a} = \frac{(21.34)(5) - (1.07)(5)(2.5) - \dfrac{(8.93)(5)(2)(5)}{(2)(3)}}{21.34} = 0.887$$

Although the preceding calculations can be easily performed by a digital computer, the graphical construction provides a visual conception of the solution.

7.9 *Continuous Girder and Frame Bridges*

Continuous girder and frame bridges can be analyzed readily by the methods of Example 7.4. In that example the application of the moment-distribution method to the solution of frames with members having variable I was illustrated. The only additional feature ordinarily encountered in the analysis of bridge frames that is not included in Example 7.4 is the calculation of stresses due to moving concentrated live loads, such as standard highway truck loading. To calculate the maximum moments and shears for such loading, it is convenient, in fact, almost necessary, to construct influence diagrams for the shear and bending moments at various sections of the frame. To fill this gap in the preceding problems, the major emphasis in this discussion will therefore be placed on the numerical procedure for constructing influence diagrams for the bending moment at the ends and at intermediate sections of continuous girders and frames. A review of the discussion in Chapter 5 of the maximum moments and shears in building frames may be desirable at this time. In addition to moving loads, some attention will be given to the calculation of internal forces produced by volumetric changes that may be caused by variation in temperature or by shrinkage after the structure is completed.

The first step in the solution is the determination of the coefficients and the construction of the influence diagrams for the fixed-end moments. These calculations can be made by the algebraic or graphical methods previously explained or, if possible, the values can be taken from curves already available, such as those prepared by the Portland Cement Association, which are given in the Appendix. It is essential, however, that a designer should be able to calculate the values, for prepared curves or tables are always limited in scope. The influence diagrams should be drawn carefully so that the ordinates at any point can be accurately scaled and also so that numerical errors may be detected.

The next step is to transform the fixed-end moments into the actual end moments, which can be accomplished almost directly by the moment-distribution method. This numerical work is greatly reduced if each fixed-end moment is treated separately and the resultant end moment then obtained from the algebraic sum of the separate values. To illustrate this procedure let us consider the three-span girder (Fig. 7.16) for which the distribution and carry-over factors are recorded on the diagram. The end moments are first expressed in terms of the fixed-end moment M_{Fba} by distributing a moment M_{Fba} equal to 100 as shown in Fig. 7.16. After the moments resulting from the rotation of the joints have been recorded, the end moments can be expressed in terms of the fixed-end moment

a	b		c		d
0	.6	.4	.4	.6	0
	+100		-28		
	-60	-40	+11.2	+16.8	
		+7.8	-2.2		
	-4.7	-3.1	+.9	+1.3	
		+.6	-.2		
	-.4	-.2	+.1	+.1	
	+34.9	-34.9	-18.2	+18.2	
		+100	-28		
	-60	-40	+11.2	+16.8	
		+7.8	-2.2		
	-4.7	-3.1	+.9	+1.3	
		+.6	-.2		
	-.4	-.2	+.1	+.1	
	-65.1	+65.1	-18.2	+18.2	

(Carry-over factor .7)

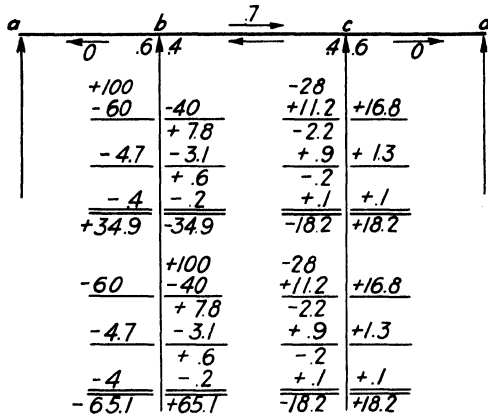



Fig. 7.16

M_{Fba} by the following equations:

$$M_{ba} = 0.349M_{Fba} \qquad M_{cd} = 0.182M_{Fba}$$

In the same manner, by distributing a fixed-end moment M_{Fbc} equal to 100, the end moments can be expressed in terms of M_{Fbc}. By taking advantage of the symmetry of the girder, the complete expression for the end moment M_{ba} becomes

$$M_{ba} = 0.349M_{Fba} - 0.651M_{Fbc} + 0.182(M_{Fcb} + M_{Fcd}) \qquad (7.12)$$

The numerical values of the fixed-end moments should be substituted into the equations for the end moments with their proper sign, that is, positive when clockwise. By means of equation 7.12, an influence diagram for M_{ba} can be constructed directly from the influence diagrams for the fixed-end moments.

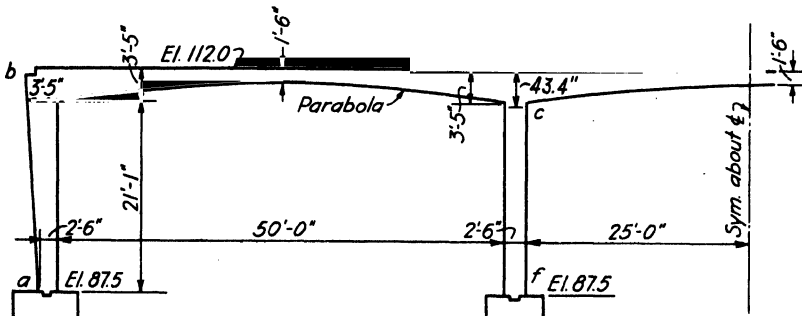



Fig. 7.17

Example 7.6 Influence diagrams for the bending moments at several sections of a three-span continuous reinforced concrete frame bridge (Fig. 7.17) will be constructed. The coefficients for the girders and abutments can be obtained from the diagrams in the Appendix; those for the piers can be calculated as was done in Example 7.4. The coefficients, as taken from the diagrams, are the following.

Girders

$$\frac{\min d}{\max d} = \frac{18}{43.4} = 0.415$$

for which

$$C_1 = C_3 = k = 16.1 \qquad \frac{C_2}{C_1} = C = 0.745$$

Abutments

$$\frac{\min d}{\max d} = \frac{30}{42} = 0.715$$

for which

$$C_1 = 5.2 \qquad C_3 = 8.6 \qquad C = \frac{C_2}{C_1} = 0.65$$

or

$$C_2 = (0.65)(5.2) = 3.38$$

If the abutment is assumed to be hinged at the top of the footing, the coefficient for the top of the member is

$$C' = C_3 - \frac{C_2{}^2}{C_1} = 8.6 - \frac{(3.38)^2}{5.2} = 6.4$$

Piers

If a value of I equal to ∞ is assumed for a distance of $0.07h$ at the top of the pier, the coefficients are

$$C_1 = 4.32 \qquad C_2 = 2.65 \qquad C_3 = 5.35$$

For the pier hinged at the top of the footing, the coefficient at the girder end is

$$C' = 5.35 - \frac{(2.65)^2}{4.32} = 3.72$$

The values of the distribution factors which are calculated for rectangular sections with I proportional to d^3 are recorded in Table 7.2.

The influence diagram (Fig. 7.18) for the fixed-end moment M_{Fbc} for the girder bc was drawn from the P.C.A. curves given in the Appendix. Since the three girders are identical, all fixed-end moments can be obtained

Table 7.2

Joint	Member	C	K_0	CK_0	$r = \frac{CK_0}{\Sigma CK_0}$
b	ba	6.4	$\frac{2.5^3}{22.9}$	4.36	0.808
	bc	16.1	$\frac{1.5^3}{52.5}$	1.04	0.192
				$\Sigma CK_0 = 5.40$	1.000
c	cb	16.1	0.0642	1.04	0.225
	cd	16.1	0.0642	1.04	0.225
	cf	3.72	$\frac{2.5^3}{22.9}$	2.53	0.550
				$\Sigma CK_0 = 4.61$	1.000

from this one diagram. To obtain the end moments in terms of the several fixed-end moments, the procedure already explained will be used. In Fig. 7.19 the end moments have been determined for a value of 100 for each fixed-end moment by the moment-distribution method. Because of symmetry, the distribution need be made for only three of the six fixed-end moments. It is assumed, of course, that no translation of the joints occurs during these calculations and therefore an auxiliary force F must be used to prevent sidesway when the vertical loads are applied. For the assumption of no sidesway, the moment M_{bc} can therefore be expressed by the equation

$$M_{bc} = 0.829M_{Fbc} - 0.143(M_{Fcb} + M_{Fcd}) + 0.024(M_{Fdc} + M_{Fde}) - 0.0034M_{Fed} \quad (7.13)$$

and the other end moments can be expressed by similar equations

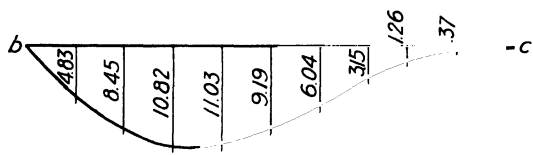
In the preceding solution an auxiliary force F prevented any horizontal movement of the deck, but whether such a force will actually be developed is difficult to predict. Some horizontal resistance is undoubtedly provided

Influence Diagram for Fixed-End Moment M_{Fbc}

Fig. 7.18

b	c	c	d	d	e
+100	-14.3		+2.4		-.40
-19.2	+3.2	+3.2	-.54	-.54	+.08
+2.4	-.34	-.40	+.13	+.06	-.03
-.46	+.17	+.17	-.04	-.04	+.01
+.13	—	—	—	—	—
-.02					
+82.85 (+67.6)	(-24.3) -11.27	+2.97 (-6.57)	(-7.6) +1.95	-.52 (-13.6)	(-15.6) -.34
-16.8	+100		-16.8		+2.8
+3.2	-22.5	-22.5	+3.8	+3.8	-.54
-.9	+2.4	+2.8	-.9	-.4	+.22
+.2	-1.2	-1.2	+.3	+.3	-.04
-.06	+.15	+.23	-.06	-.03	—
+.01	-.08	-.08	+.02	+.02	
-14.35 (-21.6)	(+72.5) +78.77	-20.75 (-25.3)	(-18.2) -13.64	+3.69 (-2.6)	(-4.8) +2.44
		+100			
-14.35 (-21.6)	(-27.5) -21.23	+79.25 (+74.7)	(-18.2) -13.64	+3.69 (-2.6)	(-4.8) +2.44
.192	.745 → .225	.225	.745 → .225	.225	← .745 .192

(-67.6) (+21.6) (+21.6) b -82.85 +14.35 +14.35 .808 a

(+30.87) +8.3 (-47.2) -58.02 (-47.2) -58.02 .55 f

(+21.2) -1.43 (+20.8) +9.95 (+20.8) +9.95 .55 g

(+15.6) +.34 (+4.8) -2.44 (+4.8) -2.44 .808 e h

Fig. 7.19

by the earth pressure on the abutments and by the roadway slab, but the amount of this resistance is always uncertain. In a symmetrical structure the dead weight will cause no sidesway, and therefore for such a frame any error due to an assumption of no translation of the deck will be for the live load only. In unsymmetrical structures both dead and live loads will produce some sidesway, and therefore, for such conditions, it is essential that the effect of any horizontal movement of the frame be studied.

The method for correcting for sidesway, which has already been explained in Example 7.4, will be applied to this problem primarily to show the effect of such movement on the influence diagrams. It will be assumed that the top of the abutments and piers move to the left a distance $E\,\Delta$ equal to 10 units. This movement produces fixed-end moments at the top of the vertical members equal to (see Table 7.2 for CK_0 values)

$$M_{Fba} = M_{Feh} = \frac{CKE\Delta}{L} = \frac{(4.36)(10)}{22.9} = 1.9$$

$$M_{Fcf} = M_{Fdg} = \frac{(2.53)(10)}{22.9} = 1.1$$

The distribution of these fixed-end moments is recorded in Fig. 7.20, and, from the moments acting at the tops of the piers and abutments, the

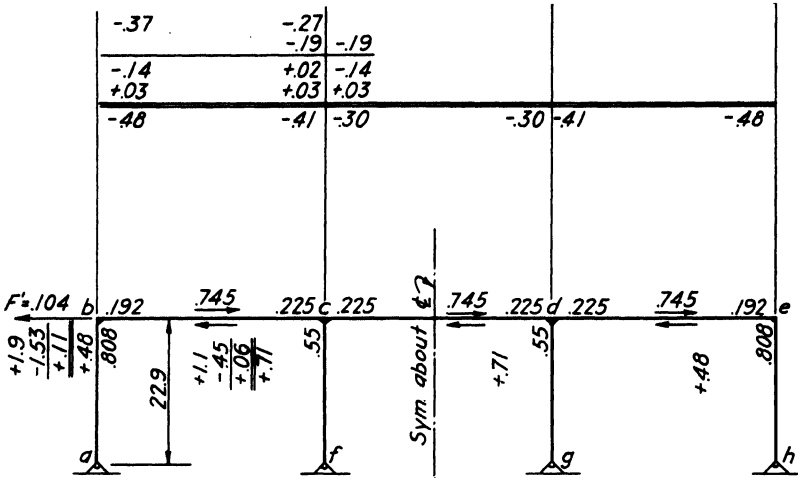



Fig. 7.20

force F' is found to be

$$F' = \frac{2.38}{22.9} = 0.104$$

From the moments recorded in Fig. 7.19, the force F required to prevent translation of the structure when a moment M_{Fbc} equal to 100 is acting is

$$F = \frac{-82.85 + 8.3 - 1.43 + 0.34}{22.9} = -\frac{75.6}{22.9} = -3.31$$

Therefore, to remove this auxiliary force F, an equal and opposite force F' must be applied which will cause end moments equal to

$$\frac{F}{F'} = \frac{3.31}{0.104} = 31.8$$

times the moments in Fig. 7.20. These corrections are added to the moments calculated in Fig. 7.19 for no sidesway, the total value being recorded in parentheses. In a similar manner, the correction for sidesway has been added to all moments calculated in Fig. 7.19.

The equation for M_{bc} in terms of the fixed-end moments after correction for sidesway becomes

$$M_{bc} = 0.676M_{Fbc} - 0.216(M_{Fcb} + M_{Fcd}) - 0.048(M_{Fdc} + M_{Fde}) - 0.156M_{Fed}$$

The coefficients of such equations for M_{bc}, M_{cb}, M_{cd}, and M_{cf} are given in Table 7.3 for both sidesway (S) and no sidesway (N.S.)

Table 7.3 Coefficients for End Moments in Terms of Fixed-End Moments with and without Sidesway

Fixed-End Moment	M_{bc} N.S.	M_{bc} S	M_{cb} N.S.	M_{cb} S	M_{cd} N.S.	M_{cd} S	M_{cf} N.S.	M_{cf} S
M_{Fbc}	0.829	0.676	−0.113	−0.243	0.030	−0.065	0.083	0.308
M_{Fcb}	−0.143	−0.216	0.788	0.725	−0.207	−0.253	−0.580	−0.473
M_{Fcd}	−0.143	−0.216	−0.212	−0.275	0.793	0.747	−0.580	−0.473
M_{Fdc}	0.024	−0.048	0.037	−0.026	−0.136	−0.182	0.100	0.208
M_{Fde}	0.024	−0.048	0.037	−0.026	−0.136	−0.182	0.100	0.208
M_{Fed}	0.003	−0.156	−0.005	−0.136	0.020	−0.076	−0.014	0.212

From the coefficients in Table 7.3 and the fixed-end moments in Fig. 7.18 the influence diagrams in Fig. 7.21 were drawn. The effect of sidesway is clearly evident from the diagrams. There is no difference in the areas of the two diagrams when the structure is symmetrical, and therefore the difference will be noticeable for the live load only.

After the influence diagrams have been constructed, the moment due to the dead weight of the structure can be calculated by considering the total weight as a number of concentrated loads. The algebraic sum of the product of each concentrated load and the ordinate to the influence

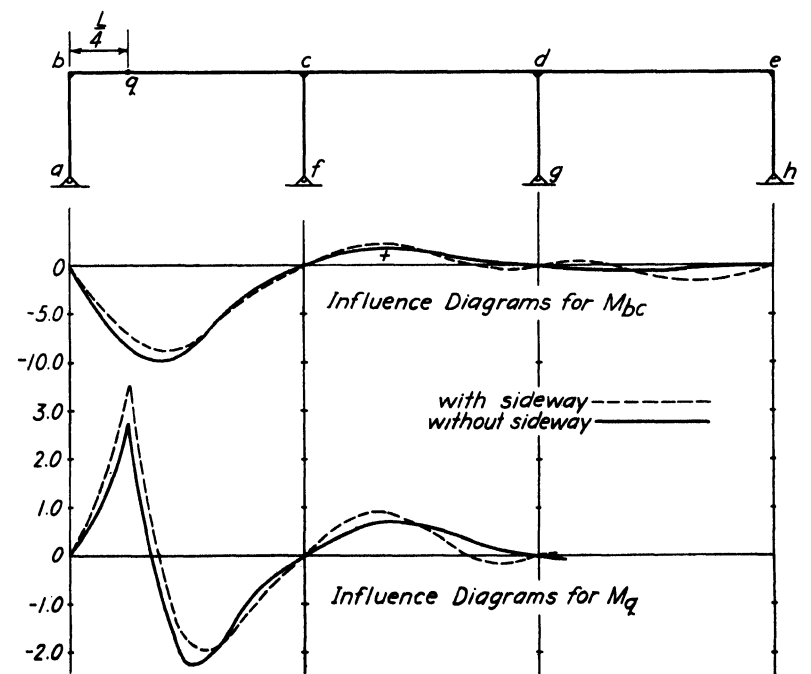



Fig. 7.21

diagram gives the dead load moment. The same procedure is used for the concentrated live load except that the position of the loads to give a maximum value must be ascertained by trial. This operation involves no particular difficulty.

In this example the curvature of the axes of the members has been neglected. When the curvature is entirely due to increase in the depth of the members, this procedure appears reasonable and gives results that are safe (see Ref. 4). When the curvature is more pronounced, the methods described in Chapter 9 are more accurate.

7.10 Moments due to Shrinkage or Temperature Change

The mathematical problem of computing the moments in continuous frames when the girders are subjected to a definite change in length will be considered in this article. It is much easier to solve this particular phase of the problem than to determine the amount of the volumetric change or the physical characteristics of the material during the movement, or to incorporate the moments into the design of the structure. These problems depend so much on the method of construction and the effect of local overstressing upon both main members and connections that no single procedure is desirable for all structures. In general, the amount of the change in length and the effect on the structure should be anticipated as accurately as possible. For reinforced concrete a nominal coefficient of shrinkage of $0.0002L$ is frequently employed, although the actual contraction will depend on many factors, such as consistency of the concrete when poured, method of construction, humidity, and temperature. Coefficients of linear expansion of 6×10^{-6} for concrete and 6.5×10^{-6} for steel structures are common values. The stresses produced by shrinkage and temperature changes depend on the structural arrangement and relative stiffness of the members. The calculation of such stresses follows the same procedure as was employed in the analysis of continuous frames subject to translation or sidesway.

After the change in length of each girder has been estimated, the relative movement of the top of each abutment or pier can be determined with respect to some point that is assumed to remain stationary. If the structure is symmetrical, the axis of symmetry, of course, provides an actual fixed point and any displacement with respect to it represents an actual movement. The problem then consists simply of distributing the fixed-end moments in the piers and abutments that are computed from the actual displacements with respect to the center line.

When the structure is unsymmetrical, some point, preferably the top of one of the piers or an abutment, must be assumed to be stationary,

and the relative motion of the other joints is determined with respect to it. This configuration of the structure requires an auxiliary horizontal force F applied at the assumed reference point to maintain equilibrium of the frame. This auxiliary force is then removed in the manner already explained. Thus, for the symmetrical single-span frame in Fig. 7.22, one-half of the total change in length Δ of the deck should be assigned to each joint with the motion toward the center. If the structure is unsymmetrical (Fig. 7.23*a*), one joint, such as *b*, should be assumed fixed and the entire movement of the deck should be assigned to joint *c*. For this condition there are fixed-end moments in member *cd* only, and an auxiliary force F is applied at *b*. This auxiliary force is then removed by applying an equal and opposite force as in Fig. 7.23*b*. The same conditions are shown for a three-span frame in Fig. 7.24. The displacements for a symmetrical frame are illustrated in Fig. 7.24*a*, and the relative displacements to be used for an unsymmetrical frame are given in Figs. 7.24*b* and *c*.

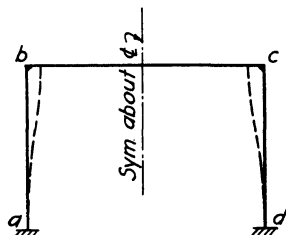



Fig. 7.22

Example 7.7 The moments in the three-span reinforced concrete frame of Fig. 7.17 will be calculated for a shrinkage of $0.0002L$ and a temperature drop of 45°F. A coefficient of linear expansion of 6×10^{-6} and a value of E of 2.5×10^6 psi are used.

The total change per unit of length of the deck is

$$0.0002 + (45)(0.000006) = 0.00047$$

Since the structure is symmetrical, the tops of the piers and abutments move towards the center with the following displacements.

Top of piers

$$(0.00047)(26.25) = 0.0123 \text{ ft}$$

Top of abutments

$$(0.00047)(78.75) = 0.037 \text{ ft}$$

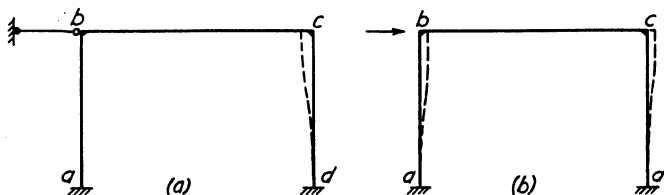



Fig. 7.23

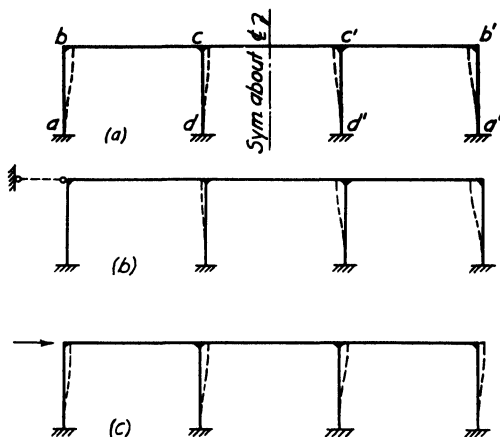



Fig. 7.24

The fixed-end moments caused by these displacements are

$$M_{Fba} = -M_{Feh} = -\frac{(4.36)(2500)(144)(0.037)}{(12)(22.9)} = -212 \text{ ft-kips}$$

$$M_{Fcf} = -M_{Fdg} = -\frac{(2.53)(2500)(114)(0.0123)}{(12)(22.9)} = -41.1 \text{ ft-kips}$$

The coefficient $\frac{1}{12}$ was restored to the moment of inertia as it was omitted from the CK_0 values in Table 7.2.

The distribution of the preceding fixed-end moments, which follows the usual procedure, is tabulated in Fig. 7.25. Only half of the structure is shown, for the corresponding moments on the right half are equal but of opposite sign. It is important to note that both end moments of the end girders are clockwise and consequently reduce the dead load moment at the abutment end but increase the value at the pier end. This distribution

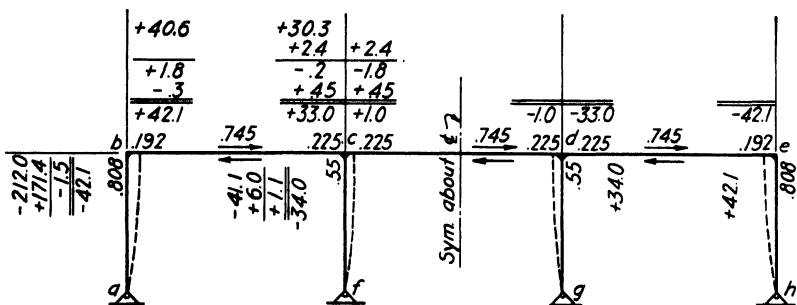



Fig. 7.25

differs from that for a single span in which the dead load moments are reduced at both ends. In contrast to this action the positive dead load bending moment at the center of the girders is increased but slightly in the three-span frame, whereas for the single span the center moment increases by the corresponding change of the end moments.

Example 7.8 The three-span frame analyzed in Example 7.7 by utilizing the symmetry of the structure will now be analyzed by the procedure required for unsymmetrical structures.

The top of the first pier, point c, will be chosen as a fixed point, so that the relative displacements of the other points with respect to it are

Point b	$(0.00047)(52.5) = 0.0247 \rightarrow$	
Point d	$(0.00047)(52.5) = 0.0247 \leftarrow$	
Point e	$(0.00047)(105.0) = 0.0494 \leftarrow$	

The fixed-end moments due to these displacements are

$$M_{Fba} = -141.7 \text{ ft-kips} \qquad M_{Fcf} = 0$$

$$M_{Fdg} = 82.2 \text{ ft-kips} \qquad M_{Feh} = 283.4 \text{ ft-kips}$$

The distribution of these fixed-end moments is recorded in Fig. 7.26, and the auxiliary force F necessary to maintain this configuration of the frame is

$$F = \frac{87.1}{22.9} = 3.81 \text{ kips}$$

To remove this force, an opposite force such as F' of Fig. 7.20 must be applied, and, as the numerical value of F' must be equal to F, the moments

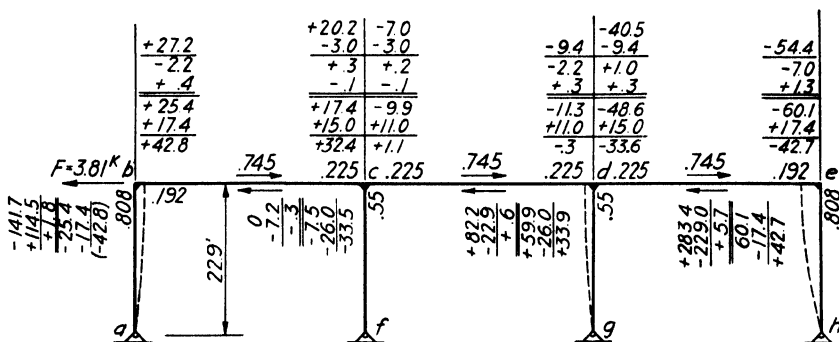



Fig. 7.26

in Fig. 7.20 must be multiplied by

$$\frac{F}{F'} = \frac{3.81}{0.104} = 36.6$$

and added to the moments in Fig. 7.26. These values, as well as the final results, are shown in Fig. 7.26. The discrepancies between the final values in Figs. 7.25 and 7.26 are due to differences in the rate of convergence and order of distribution of the two methods as two cycles of distribution were used in each problem.

7.11 *Vierendeel Trusses with Variable I*

The panel method previously described in Chapter 6 can be used for the solution of Vierendeel trusses with haunched members provided that the relative stiffnesses of members at both upper and lower panel points are the same in any panel.

If the stiffness of each member is indicated by CK_0 instead of K and if $C_3 = C_1$ so that

$$r = \frac{C_1K_0\,(\text{chord})}{C_1K_0\,(\text{vertical } h_1)}, \qquad s = \frac{C_1K_0\,(\text{chord})}{C_1K_0\,(\text{vertical } h_2)}$$

$$\alpha = \frac{h_2 - h_1}{h_1}$$

then

$$D = 2[2 + r + s + \alpha(2 + \alpha + 2s + \alpha s) + n(2 + 2\alpha - r - s - 2\alpha s - \alpha^2 s)] \quad (7.14)$$

In this expression for D, the coefficient n represents the carry-over factor C_2/C_1 which is assumed to be constant for all members in the panel. If C_2 and C_1 are 4 and 2 respectively, then $n = \frac{1}{2}$ and the value of D becomes the same as in Article 6.7. An average value of n for the chord and vertical members in any panel can be used without modifying the results appreciably.

For these conditions the equations for the primary moments take the following values:

$$M_{ad}' = M_{bc}' = \left(\frac{\alpha M - Vl}{D}\right)[1 + s + \alpha + \alpha s + n(1 - s - \alpha s)] \quad (7.15a)$$

$$M_{da}' = M_{cb}' = \left(\frac{\alpha M - Vl}{D}\right)[1 + r + n(1 - r + \alpha)] \quad (7.15b)$$

The secondary moments become

$$M_{ad}'' = M_{bc}'' = +\frac{2r(1-n)}{D}M_1 \tag{7.16a}$$

$$M_{da}'' = M_{cb}'' = -\frac{2r(1-n)(1+\alpha)}{D}M_1 \tag{7.16b}$$

and

$$M_{ad}'' = M_{bc}'' = -\frac{2s(1-n)(1+\alpha)}{D}M_2 \tag{7.17a}$$

$$M_{da}'' = M_{cb}'' = +\frac{2s(1-n)(1+\alpha)^2}{D}M_2 \tag{7.17b}$$

When the preceding equations are applied to each panel, the solution can be carried out as explained in Chapter 6.

Problems

7.9 (*a*) Construct influence diagrams for M_{FAB} and M_{FBA} for the beam in Problem 7.1 if the beam is fixed at both ends. Use the graphical solution first and then check by a numerical method. Compare the results obtained from assuming a linear variation between points on the M/I diagram with the Newmark method.

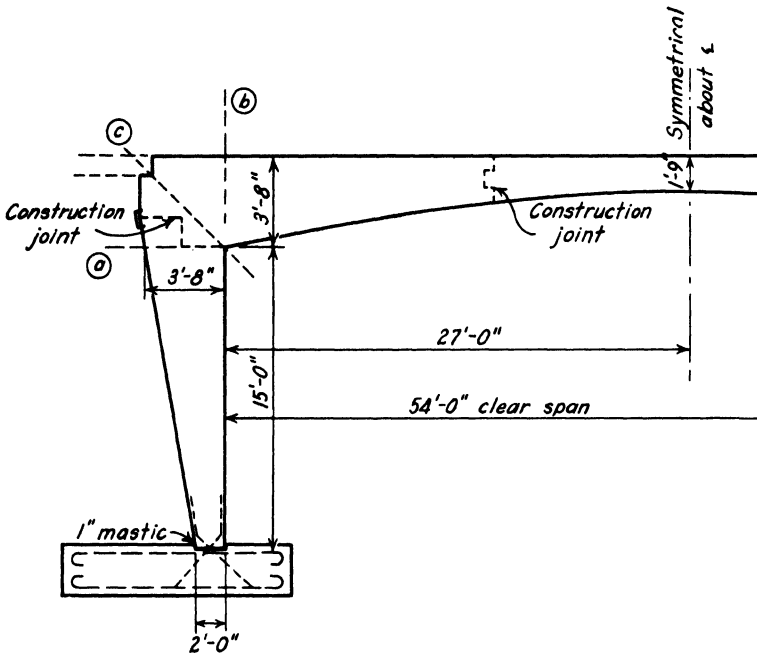
(*b*) Construct an influence diagram for M'_{FBA} if the beam is simply supported at A and fixed at B.

7.10 (*a*) Express the ordinates of the influence diagram for M_{ba} of the girder in Problem 7.4 in terms of the ordinates of the influence diagrams for the various fixed-end moments.

(*b*) Repeat part *a* for $M_{bb'}$.

7.11 Submit calculations and drawings for the partial design of the single-span rigid-frame highway bridge shown in accordance with the AASHO Specifications. The extent of the design will be designated by the instructor. An investigation of Sections *a*, *b*, and *c* should be made.

The design should be made for a typical 1 ft width of bridge for three traffic lanes of 10 ft each. An impact factor of 30% may be used together with the specified reduction when all traffic lanes are loaded simultaneously. An allowance of 40 lb per square foot of roadway should be made for a bituminous wearing surface. The bridge should be designed to withstand a temperature rise of 40°F and a fall of 60°F with a coefficient of expansion of 6×10^{-6}. An allowance for shrinkage is included in these values, which are considered adequate for design purposes.

The effect of earth pressure on the abutments should be given consideration. Assume an equivalent fluid pressure of 40 lb per cubic foot with a 2 ft surcharge for live load. The abutments can be assumed free to rotate at the base.

Problem 7.11

7.12 Prepare a computer program that will give the ordinates of the influence diagrams for the fixed-end moments for any beam with variable moment of inertia. Check your program by solving Problem 7.9. Assume that M/I values are provided as input data.

7.13 Extend the program in Problem 7.12 to give any end moment in terms of the ordinates of the fixed-end moments. Assume that the coefficients by which the fixed-end moments are multiplied will be provided as input data.

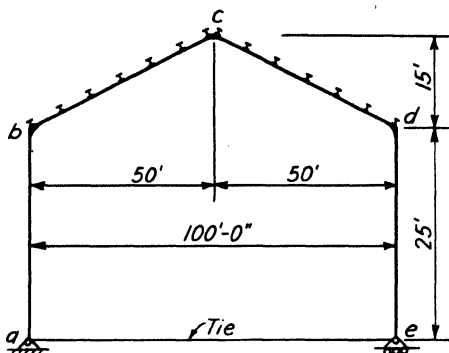
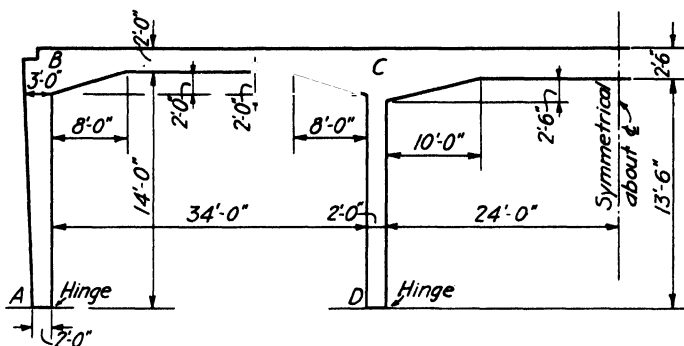
7.14 (*a*) Design the frame shown for the following loads:

Dead load of roof	20 lb per square foot
Estimated dead load of frame and bracing	8 lb per square foot
Snow load	25 lb per square foot
Wind load	25 lb per square foot on a vertical surface

Spacing of frames is 20 ft 0 in. center to center.

Use rolled steel sections with increased depth at *b*, *c*, *d* (AISC specifications).

(*b*) Show sketches of details at *a*, *b*, and *c*, and also plan of bracing. Use the diagrams in the Appendix for the preliminary design.

7.15 Construct an influence diagram for the moment at *B* and at the center of span *BC* for the continuous frame shown. Assume a width of 12 in. for all members.

Problem 7.14

Problem 7.15

7.16 Calculate the end moments for all members of the continuous frame in Problem 7.15 for a temperature drop of 60°. Use a coefficient of linear expansion of 6×10^{-6} and a modulus of elasticity of 2.5×10^{6} lb per square inch.

References

1 L. T. Evans, *Rigid Frames*, Edwards Bros.

2 A. G. Hayden, *The Rigid Frame Bridge*, John Wiley and Sons.

3 "Concrete Beams and Columns with Variable Moment of Inertia," *Portland Cement Assoc., Bull. S. T.* 41.

4 "Continuous Concrete Bridges," Portland Cement Assoc., Chicago, Illinois.

5 W. H. Weiskopf and J W. Pickworth, "Tapered Structural Members: An Analytical Treatment," *Trans. Am. Soc. C. E.*, Vol. 102 (1937).

6 D. B. Hall, "Deflections by Geometry," *Trans. Am. Soc. C. E.*, Vol. 103 (1938).

7 W. J. Eney, "Fixed-End Moments by Cardboard Models," *Eng. News-Record*, December 1935.

8 W. R. Osgood, "A Theory of Flexure for Beams with Nonparallel Extreme Fibres," *J. Appl. Mechanics*, September 1939.

9 Otakar Ondra, "Moment-Distribution Constants from Cardboard Analogs," *J. of the Struct. Div., Am. Soc. C. E.*, January 1961.

8

Continuous Trusses and Bents

8.1 Truss Deflections

Both algebraic and graphical methods that are frequently used for calculating truss deflections were explained in Chapters 2 and 3. The algebraic methods include: (1) equality of external and internal work, together with the principle of virtual work (Article 2.8); (2) Castigliano's theorem (Articles 2.10 to 2.12). Since the equations and numerical operations for these two methods are identical in their final form, either can be used in the subsequent analysis. However, the author believes that, although the principle of virtual work is more direct when the determination of unknown displacements for known force systems is involved, Castigliano's theorem has more simplicity and convenience when redundant forces must be determined for specified strain conditions.

The graphical solution most frequently employed is the application of the Williot diagram for determining relative displacements (Article 3.2) together with the Mohr rotation diagram to give absolute displacements (Article 3.3). These diagrams have the advantage of giving all displacements in one solution, whereas the algebraic methods require the calculation of each displacement separately. However, as previously shown in Article 2.8, when the principle of virtual work is used in its most general form, the calculations for displacements are greatly reduced and the

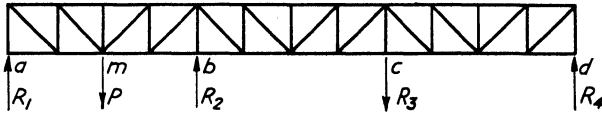



Fig. 8.1

algebraic and graphical methods are then quite comparable with respect to the time involved.

8.2 Algebraic Procedure

The equations for calculating the deflections in any truss, such as in Fig. 8.1, are readily obtained by Castigliano's theorem. The force system is replaced for analytical purposes by the three separate force systems in Figs. 8.2*a*, *b*, and *c*. If the stresses due to these force systems are combined, the total stress in any member is equal to

$$S = S' + R_1u_1 + R_2u_2 \tag{8.1}$$

where S' = the stress due to load P, Fig. 8.2*a*
u_1 = the stress due to R_1 equal to unity (Fig. 8.2*b*)
u_2 = the stress due to R_2 equal to unity (Fig. 8.2*c*)

The total strain energy in the truss is therefore equal to

$$U = \sum \frac{S^2L}{2AE} = \sum \frac{(S' + R_1u_1 + R_2u_2)^2L}{2AE} \tag{8.2}$$

By equations 2.39 and 8.2, the displacements at any point in a truss

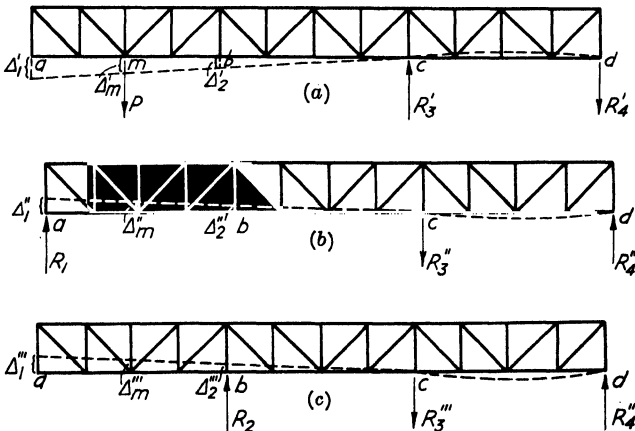



Fig. 8.2

can be calculated. Thus the displacement Δ_1 in the direction of R_1 is equal to

$$\Delta_1 = \frac{\partial U}{\partial R_1} = \sum \frac{(S' + R_1u_1 + R_2u_2)u_1L}{AE} = \sum \frac{Su_1L}{AE} \tag{8.3a}$$

and, in the same manner,

$$\Delta_2 = \frac{\partial U}{\partial R_2} = \sum \frac{(S' + R_1u_1 + R_2u_2)u_2L}{AE} = \sum \frac{Su_2L}{AE} \tag{8.3b}$$

$$\Delta_m = \frac{\partial U}{\partial P} = \sum \frac{(S' + R_1u_1 + R_2u_2)u'L}{AE} = \sum \frac{Su'L}{AE} \tag{8.3c}$$

In equation 8.3*c*, u' is the stress in the member of the simply supported truss for P equal to unity, and with R_1 and R_2 regarded as part of the applied forces. Equation 8.3*c* can also be written so as to regard R_1 and R_2 as functions of P, that is,

$$\Delta_m = \frac{\partial U}{\partial P} = \sum \frac{S(\partial S/\partial P)L}{AE} = \sum \frac{SuL}{AE} \tag{8.3d}$$

where u is the stress in the member of the continuous truss for P equal to unity. Although u' is not equal to u, the summations of the terms in equations 8.3*c* and 8.3*d* are the same, since it is immaterial whether the truss is regarded as a simply supported truss subjected to applied loads of P, R_1, and R_2 or as a continuous truss subjected to the load P.

If the displacement is desired at any point at which a load is not applied, an auxiliary or virtual force P_x, applied at the point, must be included in the forces acting on the structures.

For such a combined force system, the total stress S in any member is

$$S = S' + R_1u_1 + R_2u_2 + P_xu_x \tag{8.4}$$

in which u_x is the stress due to P_x equal to unity.

The displacement Δ_x at the point where P_x is applied is

$$\Delta_x = \frac{\partial U}{\partial P_x} = \sum \frac{(S' + R_1u_1 + R_2u_2 + P_xu_x)u_xL}{AE}$$

but, since the value of P_x is actually zero,

$$\Delta_x = \sum \frac{Su_xL}{AE} \tag{8.5}$$

in which S is the stress due to the actual load and u_x is the stress due to an auxiliary load of unity applied at point x on the simply supported structure.

All these displacements can be obtained by combining the separate displacements for each force system in Fig. 8.2. For example, the displacement Δ_m is equal to the sum

$$\Delta_m = \Delta_m' + \Delta_m'' + \Delta_m''' = \sum \frac{S'u'L}{AE} + \sum \frac{R_1 u_1 u'L}{AE} + \sum \frac{R_2 u_2 u'L}{AE}$$

which is identical with equation 8.3*c*.

8.3 Redundant Reactions

The reactions of continuous trusses are frequently calculated from strain equations obtained by assuming that the supports undergo no vertical displacements. In Fig. 8.1, if the vertical displacements at R_1 and R_2 are equal to zero, equations 8.3*a* and 8.3*b* can be written in the form

$$\Delta_1 = \frac{1}{E}\left\{\sum \frac{S'u_1L}{A} + R_1 \sum \frac{u_1^2L}{A} + R_2 \sum \frac{u_1u_2L}{A}\right\} = 0 \qquad (8.6a)$$

$$\Delta_2 = \frac{1}{E}\left\{\sum \frac{S'u_2L}{A} + R_1 \sum \frac{u_1u_2L}{A} + R_2 \sum \frac{u_2^2L}{A}\right\} = 0 \qquad (8.6b)$$

If the constant terms in these equations are represented by

$$C_1 = \sum \frac{u_1^2L}{A} \qquad C_{12} = \sum \frac{u_1u_2L}{A} \qquad C_2 = \sum \frac{u_2^2L}{A}$$

$$C' = \sum \frac{S'u_1L}{A} \qquad C'' = \sum \frac{S'u_2L}{A}$$

the reactions R_1 and R_2 are given by the following expressions:

$$R_1 = \frac{C''C_{12} - C'C_2}{C_1C_2 - C_{12}^2} \qquad (8.7a)$$

$$R_2 = \frac{C'C_{12} - C''C_1}{C_1C_2 - C_{12}^2} \qquad (8.7b)$$

Similar equations can be established for a continuous truss of any number of spans. For a four-span continuous truss, another redundant reaction, say R_3, must be used, and the following additional constant terms computed:

$$C_3 = \sum \frac{u_3^2L}{A} \qquad C_{13} = \sum \frac{u_1u_3L}{A} \qquad C_{23} = \sum \frac{u_2u_3L}{A}$$

$$C''' = \sum \frac{S'u_3L}{}$$

For a continuous truss of four spans that has no vertical displacements at the supports the following equations can be written:

$$\Delta_1 = C_1R_1 + C_{12}R_2 + C_{13}R_3 + C' = 0 \qquad (8.8a)$$

$$\Delta_2 = C_{12}R_1 + C_2R_2 + C_{23}R_3 + C'' = 0 \qquad (8.8b)$$

$$\Delta_3 = C_{13}R_1 + C_{23}R_2 + C_3R_3 + C''' = 0 \qquad (8.8c)$$

from which the values of R_1, R_2, and R_3 can be computed.

8.4 *Influence Diagrams for Reactions*

The Müller-Breslau method for constructing influence lines for redundant forces, as explained in Article 7.8, can also be used to advantage in the analysis of continuous trusses. This method, which is based on the reciprocal theorem, requires the determination of the ordinates to an elastic curve which is caused by applying a special virtual or auxiliary force system. The deflections of the truss can be determined algebraically by the principle of virtual work or graphically by use of the Williot-Mohr vector diagrams.

Thus, if the influence diagram for the reaction R_1 in Fig. 8.1 for a unit vertical load is desired, it can be obtained from the elastic curve for the force system shown in Fig. 8.3. This force system consists of an applied force R_1' in the assumed direction of the redundant R_1 and restraining forces R_2', R_3', R_4', which will prevent displacements at R_2, R_3, and R_4 respectively. Therefore, when the reciprocal theorem is applied to the two force systems in Figs. 8.1 and 8.3, the following equation is obtained:

$$R_1\,\Delta_1 - Py = 0 \qquad (8.9a)$$

since the displacements in the direction of all other forces are zero.

From equation 8.9*a* the value of the redundant reaction R_1 for a unit value of P is given by the expression

$$R_1 = (1)\,\frac{y}{\Delta_1} \qquad (8.9b)$$

The construction of the influence diagram therefore requires the following steps:

1. Apply R_1' equal to any value, say unity, and calculate R_2', R_3', R_4'. Because one of these values is a redundant, it must be determined

Fig. 8.3

in the usual manner from equation 8.6*b*. Thus, if R_2' is selected as the redundant,

$$\Delta_2 = R_1' \sum \frac{u_1 u_2 L}{A} + R_2' \sum \frac{u_2^2 L}{A} = 0$$

or

$$R_2' = -R_1' \frac{\sum \frac{u_1 u_2 L}{A}}{\sum \frac{u_2^2 L}{A}} \tag{8.10}$$

in which the u stresses have the same meaning as previously described.

2. From the values of R_1', R_2', R_3' and R_4', calculate the change in length of each member SL/AE. If constant, E can be assumed equal to unity. The values of SL/A should be tabulated on a diagram of the truss.
3. Calculate the values of y and Δ_1, either algebraically or graphically, and determine the value of R_1 from equation 8.9*b*.

Example 8.1 The ordinates to the influence diagram for the reaction R_1 of the three-span continuous truss in Fig. 8.4 will be calculated by the Müller-Breslau method. The lengths and areas of the members are recorded on the truss diagram. Values of u_1 and u_1L/A for R_1 equal to unity and u_2 and u_2L/A for R_2 equal to unity are given in Figs. 8.5*a* and *b*, respectively. By means of equation 8.10, the reaction R_2' is obtained directly from the u_1, u_2, and u_2L/A quantities for any applied force R_1'. The summations of these terms for all members of the truss are

$$\sum \frac{u_1 u_2 L}{A} = 347.5 \qquad \sum \frac{u_2^2 L}{A} = 222.7$$

Substituting these quantities in equation 8.10 gives the value of R_2' as

$$R_2' = -\frac{347.5}{222.7} R_1' = -1.56 R_1'$$

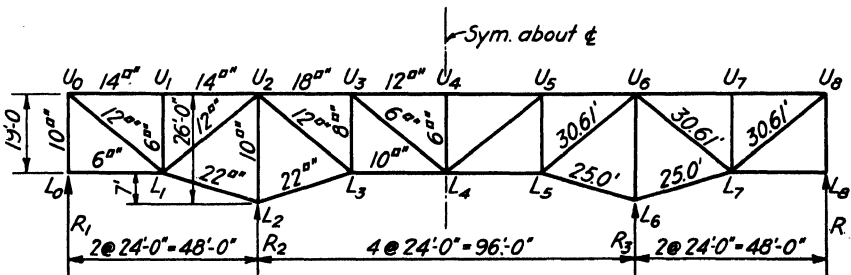



Fig. 8.4

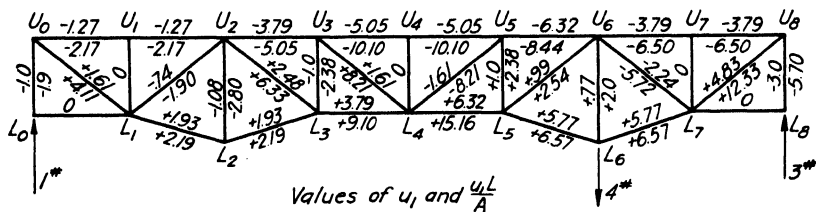
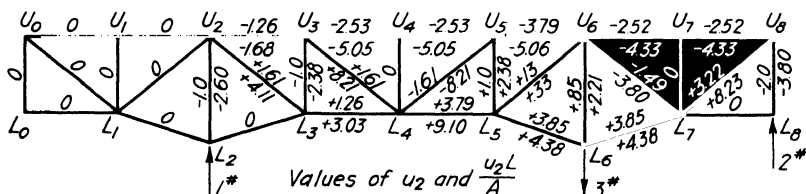
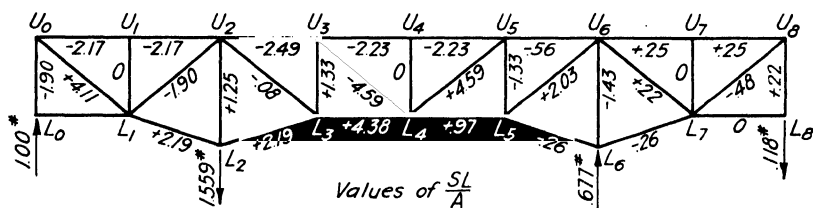



Fig. 8.5

For an applied load R_1' equal to unity the reactions that will prevent any vertical displacement at L_2, L_6, and L_8 and also satisfy the equilibrium requirements are

$$R_2' = -1.56 \qquad R_3' = 0.677 \qquad R_4' = -0.118$$

The stresses S in the truss members for R_1' equal to unity are therefore equal to

$$S = u_1 - 1.56u_2$$

and the change in length of each member (times E) is

$$\frac{SL}{A} = \frac{u_1L}{A} - 1.56\,\frac{u_2L}{A}$$

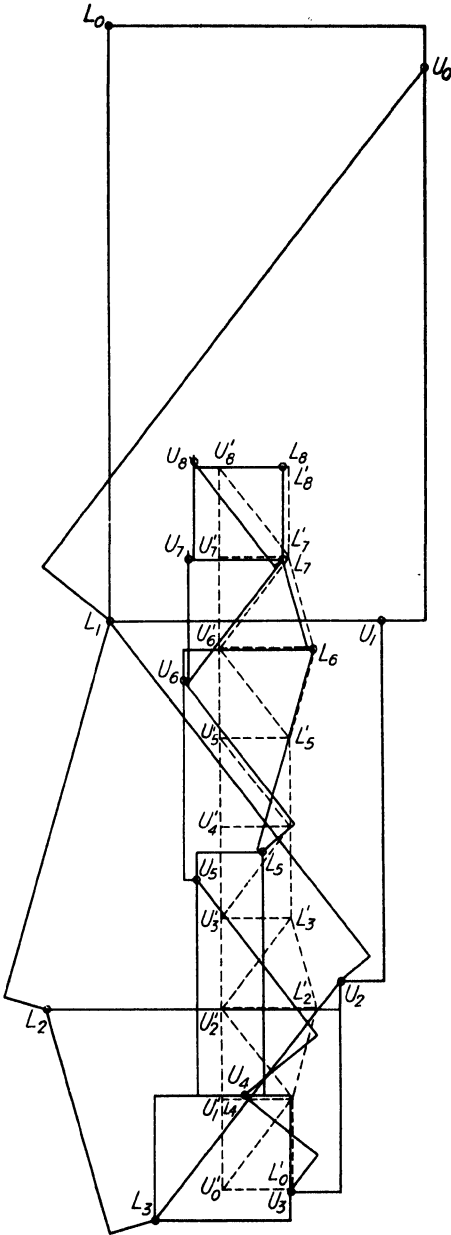
The numerical values of SL/A as recorded on the truss diagram in Fig. 8.5c were used to draw the Williot diagram in Fig. 8.6. In this diagram L_4 was selected as the fixed point and member U_4L_4 as a fixed axis. The construction of the displacement diagram which follows the procedure explained in Article 3.2 should be executed accurately on a

Displacement Diagram
For A Continuous Truss

Fig. 8.6

large drawing board with good equipment. The position of each point should be backchecked as soon as it is determined in order to avoid extending any error to succeeding points.

After the Williot diagram is completed, it is apparent that U_4L_4 is not a fixed axis, for such an assumption gives a relative vertical displacement of points L_2, L_6, and L_8. Consequently, the structure is rotated about L_6 as a fixed point until the displacement at L_2 is zero. The displacement vectors due to the rotation are given by the Mohr rotation diagram, which is indicated by broken lines. Since the displacement of L_8 must also be zero, a check on the accuracy of the drawing work is obtained. The vertical displacement of any point is equal to the vertical distance from the point on the Mohr diagram to the corresponding point on the Williot diagram. The vertical movement of L_0 is equal to the vertical component of the vector $L_0'L_0$ or 55.0 units, and the vertical displacement of U_1 is 22.5 units. The value of R_1 in Fig. 8.4 for a unit load at U_1 is therefore

$$R_1 = \frac{y}{\Delta_1} = \frac{22.5}{55.0} = 0.409$$

The vertical displacement of all upper panel points and the corresponding ordinates to the influence diagram are tabulated in Table 8.1. The influence diagram in Fig. 8.7 is drawn from these values.

Table 8.1

Point	Vertical Displacement y	Value of R_1 $\frac{y}{55.0}$
U_0	+53.1	+0.965
U_1	+22.5	+0.409
U_2	+1.2	+0.022
U_3	−12.9	−0.235
U_4	−12.6	−0.229
U_5	−6.6	−0.120
U_6	−1.4	−0.025
U_7	−0.1	−0.002
U_8	+0.2	+0.004

A check on these values is obtained by computing the displacement of L_0 algebraically, which gives

$$\Delta_1 = \frac{1}{E}\sum u_1 \frac{SL}{A} = \frac{55.0}{E}$$

which is identical with the value obtained from the Williot diagram.

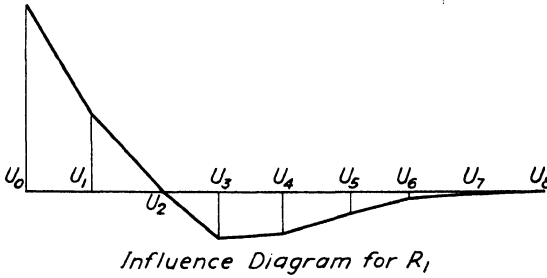



Fig. 8.7

Example 8.2 The vertical displacements due to the auxiliary force system in Fig. 8.5*c* which were obtained graphically in Example 8.1 can also be determined algebraically by the principle of virtual work. Thus, if the vertical displacements of points L_2 and L_3 are first assumed to be zero, then the relative vertical displacement of point U_4 can be calculated from the virtual forces and stresses shown in Fig. 8.8*a*, together with the SL/A values recorded on the diagram in Fig. 8.5*c*. Equating the corresponding external and internal work performed by the virtual forces u in Fig. 8.8*a* acting through the displacements SL/A in Fig. 8.5*c* gives the relation

$$(1)y_4' + (2)(0) + (1)(0) = \sum u \frac{SL}{A} = +16.00$$

Therefore the relative vertical displacement y_4' of point U_4 is

$$y_4' = +16.00$$

Using the virtual forces u in Fig. 8.8*b*, the relative vertical displacement y_5' of point U_5 can be calculated from the equation

$$(1)y_5' - (2)(16.00) + (1)(0) = \sum u \frac{SL}{A} = +4.29$$

or

$$y_5' = +36.29$$

From the virtual forces in Fig. 8.8*c*, we obtain the relative vertical displacement y_{L6}' of point L_6

$$(1)(y_{L6}') - (2)(36.29) + (1)(16.00) = \sum u \frac{SL}{A} = 0.57$$

from which

$$y_{L6}' = +57.15$$

However, since the actual vertical displacements of both L_2 and L_6 are zero, the truss must be rotated about point L_2 so that point L_6 moves

U2 -1.26
U3
U4
u = +1.61
-1.61
+1.00
-1.00
u = +1.26
L2
L3
L4
(a)
U3 -1.26
U4 -1.26
U5
+1.61
(b)
U4
U5
U6
-1.26
-1.61
+1.26
L5
L6
(c)

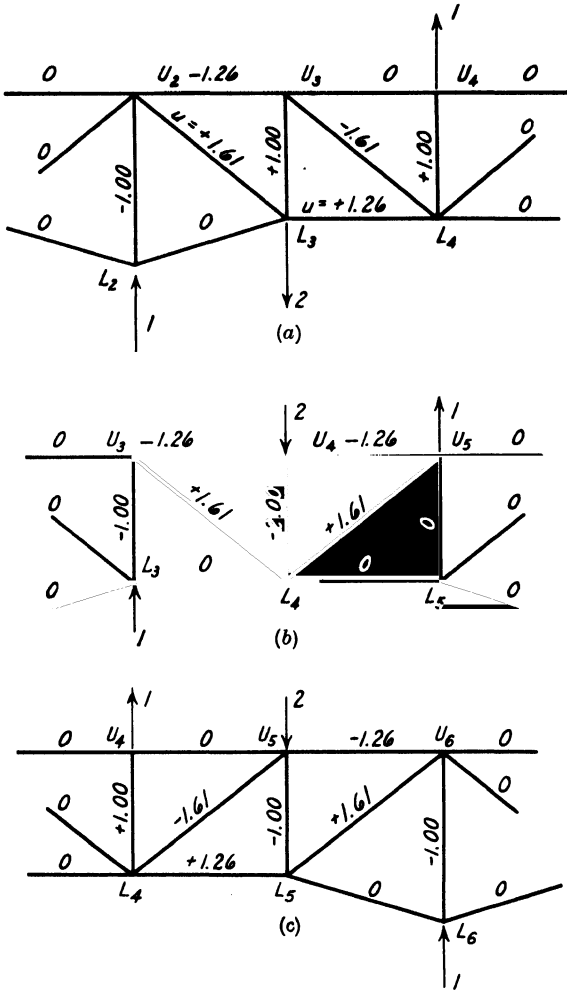



Fig. 8.8

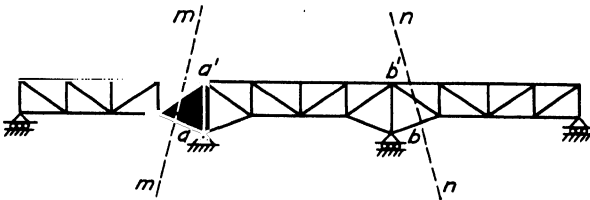



Fig. 8.9

vertically downward an amount of -57.15. The actual displacements of U_2, U_3, U_4, U_5, and U_6 can now be determined as follows:

$$y_2 = \frac{SL}{A} \text{ of } U_2L_2 = +1.25$$

$$y_3 = \frac{SL}{A} \text{ of } U_3L_3 - \frac{57.15}{4} = 1.33 - 14.29 = -12.96$$

$$y_4 = +16.00 - \frac{57.15}{2} = -12.58$$

$$y_5 = +36.29 - 42.86 = -6.57$$

$$y_6 = \frac{SL}{A} \text{ value of } U_6L_6 = -1.43$$

This application of the principle of virtual work when combined with the Müller-Breslau method of constructing influence diagrams for redundant forces provides a convenient solution for large truss bridges.

8.5 *End Forces and Couples*

The forces acting upon any span of a continuous truss, such as span *ab*, Fig. 8.9, are analogous to those in a continuous beam since they consist of the applied loading, end shears, and end couples. Moreover, as in the analysis of continuous beams, any span *ab*, Fig. 8.9, can first be assumed as fixed at the ends, and these fixed-end forces afterwards corrected so as to provide the necessary equilibrium and strain conditions between continuous spans. The horizontal components at a and a', also b and b', must balance, and the horizontal displacement of these points must be the same for both adjacent spans.

To illustrate the preceding statements graphically, the fixed-end forces in Fig. 8.10*a* must be corrected by adding the end forces in Figs. 8.10*b* and *c* so as to make the end rotations

$$\theta_a = \frac{\Delta_a + \Delta_{a'}}{h_a} \qquad \text{and} \qquad \theta_b = \frac{\Delta_b + \Delta_{b'}}{h_b}$$

equal for both spans. These end forces or couples can be expressed in terms of the rotation θ_a and θ_b, as was done for the end couples applied to continuous beams. All three problems represented by Figs. 8.10*a*, *b*, and *c* are statically indeterminate problems, each of which can be solved by the methods already explained. The solution of these problems gives the fixed-end couples, stiffness factors, and carry-over factors—quantities that correspond to the same terms used in the moment-distribution method for continuous beams.

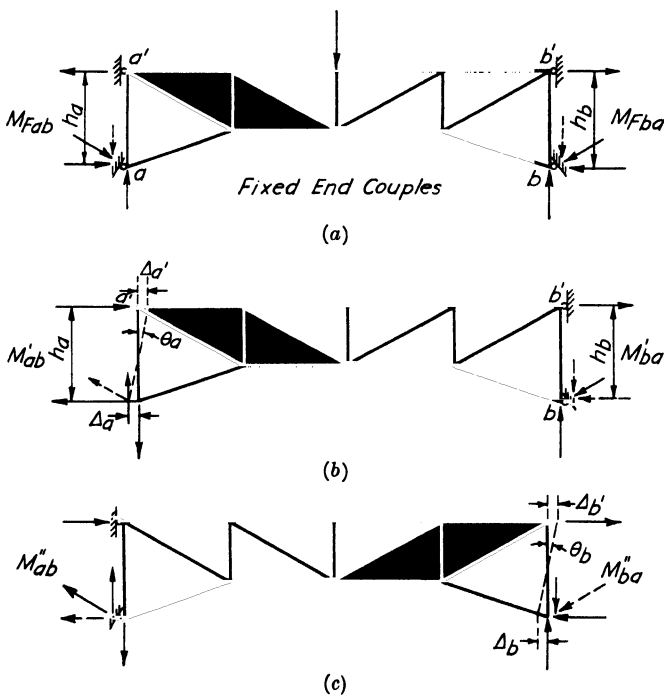



Fig. 8.10

8.6 Distribution and Carry-Over Factors

The determination of the end forces is best illustrated by calculating the end couple M_{ba}', Fig. 8.10*b*, that is required to hold joints b and b' without horizontal translation when some end couple, M_{ab}', is applied at end a. The total strain energy U in the truss for this force system is equal to

$$U = \sum \frac{(M_{ab}'u_a + M_{ba}'u_b)^2 L}{2AE} \tag{8.11}$$

where u_a equals stress for $M_{ab}' = 1$, $M_{ba}' = 0$.
u_b equals stress for $M_{ab}' = 0$, $M_{ba}' = 1$.

The end rotations θ_a and θ_b, as given by Castigliano's theorem, are

$$\theta_a = \frac{\partial U}{\partial M_{ab}'} = \sum \frac{(M_{ab}'u_a + M_{ba}'u_b)u_a L}{AE} \tag{8.12a}$$

$$\theta_b = \frac{\partial U}{\partial M_{ba}'} = \sum \frac{(M_{ab}'u_a + M_{ba}'u_b)u_b L}{AE} \tag{8.12b}$$

When θ_b is equal to zero, equation 8.12*b* gives

$$M_{ba}' = -\frac{\sum \frac{u_a u_b L}{A}}{\sum \frac{u_b^2 L}{A}} M_{ab}' = C_{ab} M_{ab}' \tag{8.13}$$

where

$$C_{ab} = -\frac{\sum \frac{u_a u_b L}{A}}{\sum \frac{u_b^2 L}{A}} \tag{8.14}$$

is the carry-over factor from a to b. Since the term $\sum \frac{u_a u_b L}{A}$ is negative, the value of C_{ab} in equation 8.14 is positive.

If the value of M_{ba}' from equation 8.13 is now substituted in equation 8.12*a*, the expression for θ_a becomes

$$\theta_a = \sum \frac{(M_{ab}' u_a + C_{ab} M_{ab}' u_b) u_a L}{AE} \tag{8.15a}$$

from which

$$M_{ab}' = \frac{E\theta_a}{\sum \frac{u_a^2 L}{A} + C_{ab} \sum \frac{u_a u_b L}{A}} = C_a E \theta_a \tag{8.15b}$$

where

$$C_a = \frac{1}{\sum \frac{u_a^2 L}{A} + C_{ab} \sum \frac{u_a u_b L}{A}} \tag{8.15c}$$

is the stiffness factor and corresponds to $C_1 K_0$ or $C_3 K_0$ for a beam.

The end couple M_{ab}'' (Fig. 8.10*c*) that will hold joints a and a' fixed when a couple M_{ba}'' is applied at end b, is calculated in a similar manner. The necessary equations are obtained by interchanging the subscripts a and b in equations 8.13, 8.14, and 8.15, or

$$M_{ab}'' = C_{ba} M_{ba}'' \tag{8.16a}$$

in which

$$C_{ba} = -\frac{\sum \frac{u_a u_b L}{A}}{\sum \frac{u_a^2 L}{A}} \tag{8.16b}$$

is the carry-over factor from b to a. Also,

$$M_{ba}'' = C_b E \theta_b \tag{8.17a}$$

where

$$C_b = \frac{1}{\sum \frac{u_b^2 L}{A} + C_{ba} \sum \frac{u_b u_a L}{A}} \tag{8.17b}$$

After the coefficients C_a and C_b have been computed for each span, the distribution factors r can be determined as

$$r_{ab} = \frac{C_a}{\sum C_a} \quad \text{and} \quad r_{ba} = \frac{C_b}{\sum C_b} \tag{8.18}$$

or the ratio of the coefficient of one span to the sum of the coefficients for both spans.

Equation 8.15c can be used when the truss is simply supported at b by making u_b equal to zero, or

$$C_a = \frac{1}{\sum \frac{u_a^2 L}{A}} \tag{8.19a}$$

When the truss is simply supported at a, the coefficient C_b becomes

$$C_b = \frac{1}{\sum \frac{u_b^2 L}{A}} \tag{8.19b}$$

8.7 *Fixed-End Couples*

Either the algebraic or preferably the semigraphic method can be used to calculate the end couples required to hold points a, a', b, b' without translation when any load is applied on the span. The stress in any member is equal to

$$S = S' + M_{Fab} u_a + M_{Fba} u_b \tag{8.20}$$

where S' = stress due to applied load on a simply supported span.
u_a, u_b = stress due to unit end couples at a and b, respectively.

The total strain energy in the truss is

$$U = \sum \frac{S^2 L}{2AE} = \sum \frac{(S' + M_{Fab} u_a + M_{Fba} u_b)^2 L}{2AE} \tag{8.21}$$

and the end rotations θ_a and θ_b equal

$$\theta_a = \frac{\partial U}{\partial M_{Fab}}$$

$$= \frac{1}{E}\left\{\sum \frac{S'u_aL}{A} + M_{Fab}\sum \frac{u_a^2L}{A} + M_{Fba}\sum \frac{u_au_bL}{A}\right\} = 0 \quad (8.22a)$$

$$\theta_b = \frac{\partial U}{\partial M_{Fba}}$$

$$= \frac{1}{E}\left\{\sum \frac{S'u_bL}{A} + M_{Fab}\sum \frac{u_au_bL}{A} + M_{Fba}\sum \frac{u_b^2L}{A}\right\} = 0 \quad (8.22b)$$

The fixed-end couples can be obtained by solving equations 8.22*a* and 8.22*b* simultaneously. These equations are similar to equations 8.6*a* and *b* except that the summation is for one span only instead of the entire structure.

Influence diagrams for the fixed-end moments or couples in a continuous truss are best obtained directly from an elastic curve of the structure by applying the Müller-Breslau principle as in Article 7.8 for the fixed-end moments in continuous beams. The procedure is similar to that followed in other problems that have been solved by the reciprocal theorem. If some end couple M_{ab}' is applied at end a (Fig. 8.10*b*), the couple M_{ba}' required to hold end b is, from equation 8.13,

$$M_{ba}' = C_{ab}M_{ab}'$$

The stresses S and SL/A, the change in length times E, must be determined for any assumed value of M_{ab}'. If M_{ab}' is taken equal to unity

$$S = u_a + C_{ab}u_b$$

$$\frac{SL}{A} = \frac{u_aL}{A} + C_{ab}\frac{u_bL}{A}$$

The displacements y and rotation θ_a are given directly by a Williot diagram drawn for the SL/A values if point b is selected as a fixed point and bb' as a fixed axis. The ordinates to the influence diagram for M_{Fab}, as in other problems, are equal to

$$M_{Fab} = P\frac{y}{\theta_a} = Ph_a\left(\frac{y}{\Delta_a + \Delta_{a'}}\right) \quad (8.23)$$

Example 8.3 The end moments and reactions of the continuous truss of Example 8.1 (Fig. 8.4) will be determined for a unit load at point U_4 by the moment-distribution method.

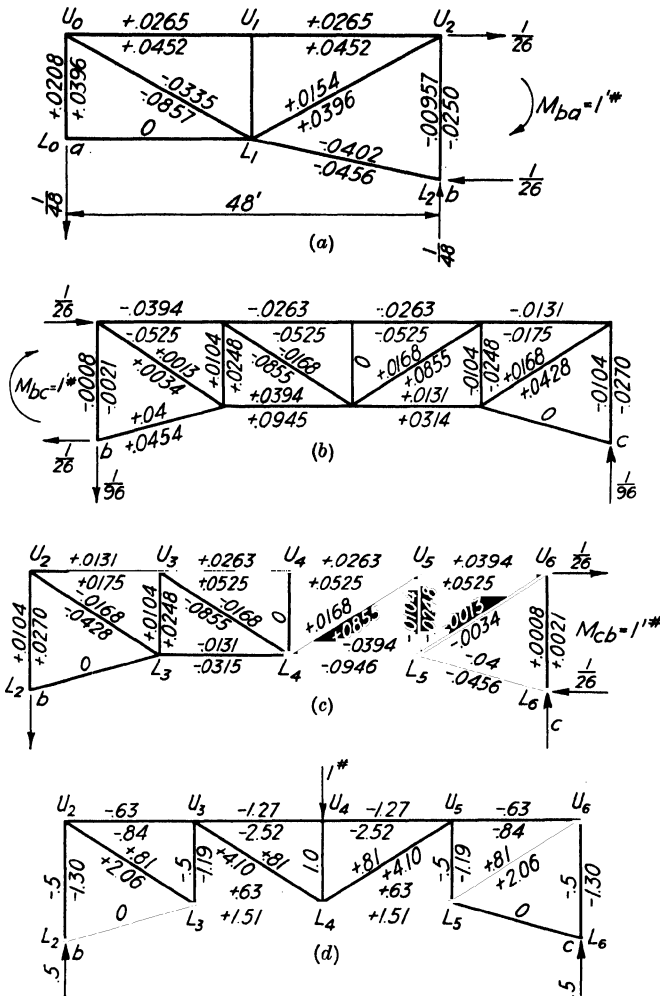



Fig. 8.11

The stiffness coefficient C_b for the end spans is given by equation 8.19*b*, for the spans are simply supported at L_0 and L_8. The values of u_b and u_bL/A for a unit end moment applied to the end span are recorded in Fig. 8.11*a*. From these values, the coefficient C_{b1} is equal to

$$C_{b1} = \frac{1}{\sum \frac{u_b^2 L}{A}} = \frac{1}{0.00877} = 114$$

The values of u_b, u_bL/A, u_c, and u_cL/A are recorded in Figs. 8.11*b* and *c* for positive unit end couples applied to the center span. From these quantities, the coefficients C_{b2}, C_{bc}, and C_{c2} are found to be

$$C_{bc} = C_{cb} = -\frac{\sum \frac{u_b u_c L}{A}}{\sum \frac{u_c^2 L}{A}} = -\frac{-0.00343}{0.0155} = 0.221$$

$$C_{b2} = C_{c2} = \frac{1}{\sum \frac{u_b^2 L}{A} + C_{bc} \sum \frac{u_b u_c L}{A}} = \frac{1}{0.0155 + (0.221)(-0.00343)} = 68$$

The distribution factors at supports *b* and *c* are proportional to the coefficients, or

For end spans *ba* or *cd*,

$$r = \frac{114}{114 + 68} = 0.626$$

For center span,

$$r = \frac{68}{182} = 0.374$$

The fixed-end moments for a unit load at U_4 will be calculated from equations 8.22*a* and *b*. The values of $S'L/A$ are recorded on the truss diagram in Fig. 8.11*d*, and from these quantities and the previous values of u_b and u_c the end rotations are found to be

$$\sum \frac{S' u_b L}{A} = 0.308 \qquad \sum \frac{S' u_c L}{A} = -0.308$$

Equations 8.22*a* and 8.22*b* give

$$\theta_b = 0.308 + M_{Fbc}(0.0155) + M_{Fcb}(-0.00343) = 0$$
$$\theta_c = -0.308 + M_{Fbc}(-0.00343) + M_{Fcb}(0.0155) = 0$$

from which

$$M_{Fbc} = -16.27 \text{ ft-lb} \qquad M_{Fcb} = +16.27 \text{ ft-lb}$$

The distribution of the fixed-end moments in Fig. 8.12*a* follows the same numerical operations as for a continuous beam. The final moment over the supports is 11.10 ft-lb, which makes the reaction R_1 equal to

$$R_1 = \frac{11.1}{48} = 0.231$$

as compared to 0.229 in Example 8.1.

A similar solution for a unit load at point U_1 is shown in Fig. 8.12*b*.

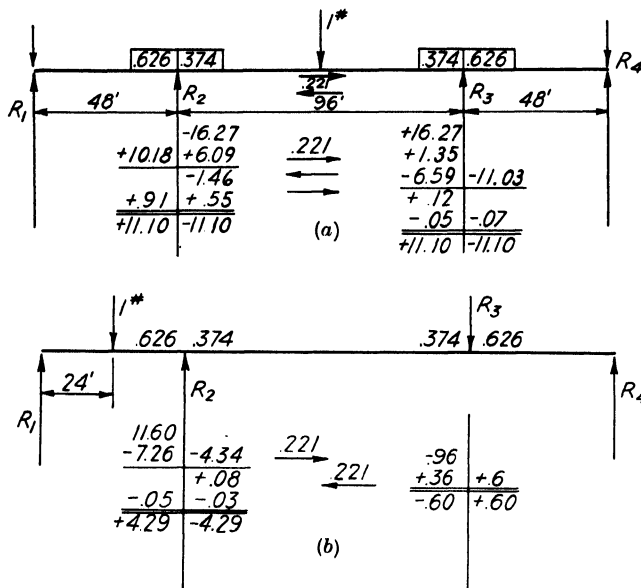



Fig. 8.12

For the moments shown, R_1 is equal to 0.411 as compared to 0.409 in Table 8.1.

8.8 Use of an Equivalent Beam

The deformation and stress in the chord member of a truss are similar to those in the flange of a beam. Some differences exist, however, as the axial stress in a truss member is always constant across a panel, whereas the flange stress in a beam ordinarily varies throughout the span. Furthermore, the deformation in the truss diagonals is usually more than the shearing deformation in a beam, so that the change in length of the diagonals reduces the carry-over and stiffness factors of a truss more than the shearing deformation does in a beam. In Chapter 4 it was shown that the end moments in a continuous beam are not appreciably affected by the shearing deformation even though the carry-over and CK values are changed considerably. For this reason the end moments in a continuous truss can often be approximately obtained from a continuous beam whose flanges give about the same relative stiffness in the various spans as do the chord members of the truss. In general, the substitution of an equivalent continuous beam for a continuous truss is a logical procedure for making a preliminary design, and usually such a preliminary design will require only slight, if any, modification.

The term equivalent beam is a descriptive term only, since there is probably no beam exactly equivalent in deformation to a truss, or at least it is not practical to define such a beam in mathematical terms. The difficulty involved in an exact transformation is indicated by comparing the angle change $\Delta\phi$ in the upper panel points of a deck truss (Fig. 8.13) with the corresponding angle change in a girder. From geometry the angle change $\Delta\phi$ at point b is

$$\Delta\phi = \frac{y_3 - y_2}{d_2} - \frac{y_2 - y_1}{d_1} = \frac{y_3 - y_2[(d_1 + d_2)/d_1] + y_1(d_2/d_1)}{d_2}$$

but, by the principle of virtual work,

$$1^k y_3 - 1^k\left(\frac{d_1 + d_2}{d_1}\right)y_2 + y_1(1^k)\left(\frac{d_2}{d_1}\right) = \sum_1^5 u\left(\frac{SL}{AE}\right)$$

in which the summation involves only the truss members 1–5 as shown in the heavy lines since the u stresses due to the virtual forces shown are zero for all other members. The actual force system (S stresses) is not indicated on the figure. Therefore for a truss the angle change at point b is

$$\Delta\phi = \frac{1}{d_2(1^k)} \sum_1^5 u\left(\frac{SL}{AE}\right) \tag{8.24a}$$

However, we know that for a girder the corresponding angle change must be placed at the centroid of the M/EI diagram and would have a value of

$$\Delta\phi = \int_{x=x_1}^{x=x_1+d_1+d_2} \frac{M\,dx}{EI} \tag{8.24b}$$

Obviously, if an exact solution is desired, equation 8.24a should be used and equation 8.24b should be considered only for approximate solutions.

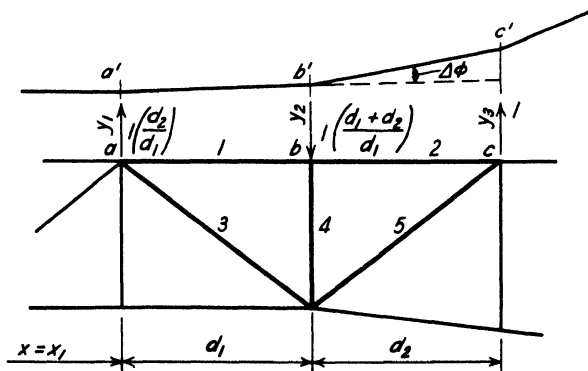



Fig. 8.13

For approximate solutions any reasonable choice of an equivalent beam may be made, such as using a beam with the same variation in moment of inertia as is given by the chord of the truss. Other approximations such as assuming that the moments of inertia vary as the cube of the depth will often give good results for preliminary design.

The differences between the fixed-end moments, distribution factors, and carry-over factors of the truss and those of the equivalent beam, although of considerable magnitude, will seldom need any consideration, for the final moments are not greatly affected by these variations. The use of an equivalent beam transforms the problem into the analysis of a continuous beam with variable moment of inertia—a problem that has already been discussed in detail in Chapter 7.

Example 8.4 The end moments in the continuous truss that was used in Examples 8.2 and 8.3 will be calculated by means of an equivalent beam. The depth of the beam will be taken as the same as that of the truss, and the moment of inertia is assumed to vary as the cube of the depth. The coefficients C_1, C_2, and C_3 and the fixed-end moments for such a beam can be selected from curves or calculated by the methods previously explained. In this problem the values are taken from the P.C.A. diagrams in the Appendix. The coefficients for the beams are the following.

End spans

$$\frac{\min d}{\max d} = \frac{19}{26} = 0.73 \qquad a = 0.5$$

$$C_1 = 4.4 \qquad C_2 = 3.0 \qquad C_3 = 7.0$$

$$C_3' = 7.0 - \frac{3.0^2}{4.4} = 5.0$$

$$C_3'K = (5.0)\frac{I_0}{L} = (5.0)\left(\frac{I_0}{48}\right) = 0.104I_0$$

Center span

$$\frac{\min d}{\max d} = 0.73 \qquad a = 0.25$$

$$C_1 = C_3 = 6.0 \qquad \frac{C_2}{C_1} = 0.60$$

$$C_1K = (6.0)\left(\frac{I_0}{L}\right) = (6.0)\frac{I_0}{96} = 0.063I_0$$

Distribution factors at b and c are

$$r_{ba} = r_{cd} = \frac{0.104I_0}{0.104 \;{}_0 + 0.063I_0} = 0.625$$

$$r_{bc} = r_{cb} = \frac{0.063I_0}{0.104I_0 + 0.063I_0} = 0.375$$

Fixed-end moments

For a unit load at U_4

$$M_{Fbc} = -(0.143)(1)(96) = -13.8 \text{ ft-lb}$$

$$M_{Fcb} = +13.8 \text{ ft-lb}$$

For a unit load at U_1

$$M_{Fab} = (-0.105)(1)(48) = -5.04 \text{ ft-lb}$$

$$M_{Fba} = (+0.169)(1)(48) = +8.10 \text{ ft-lb}$$

When $\qquad M_{ab} = 0, \quad M_{Fba} = (8.10) + \left(\frac{3.0}{4.4}\right)5.04 = 11.54 \text{ft-lb}$

The distribution of these fixed-end moments together with the final moments is shown in Figs. 8.14*a* and *b*. The corresponding values that were obtained in Example 8.3 for the continuous truss are given in parentheses. For the unit load at U_4, the midpoint of the center span,

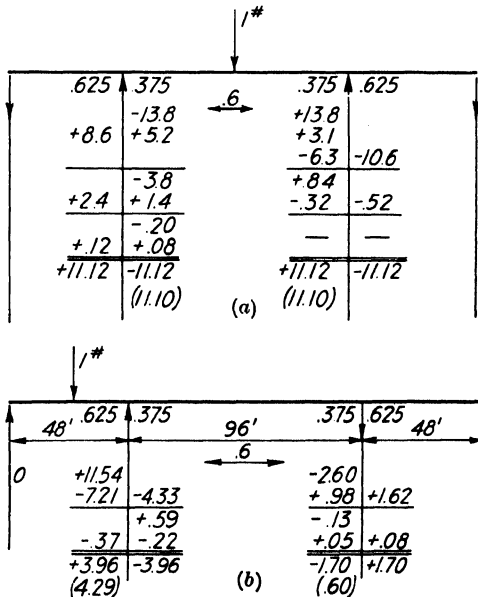



Fig. 8.14

these end moments are practically identical in spite of the great difference between the fixed-end moments and carry-over factors for the two structures.

For a unit load at U_1 the moment over R_1 is about the same as for the continuous truss in Example 8.2, but over R_3 the moment for the beam is considerably larger than for the truss. This difference is mainly due to the short spans, which tend to increase the shearing effect. However, even for a truss of these proportions, the moments obtained by the continuous beam give a fair approximation to the actual values; at least, they are sufficiently accurate for a preliminary design.

8.9 *Continuous Bents*

When horizontal forces are applied to a truss that is supported by columns (Fig. 8.15*a*), such forces are carried to the ground by the shears

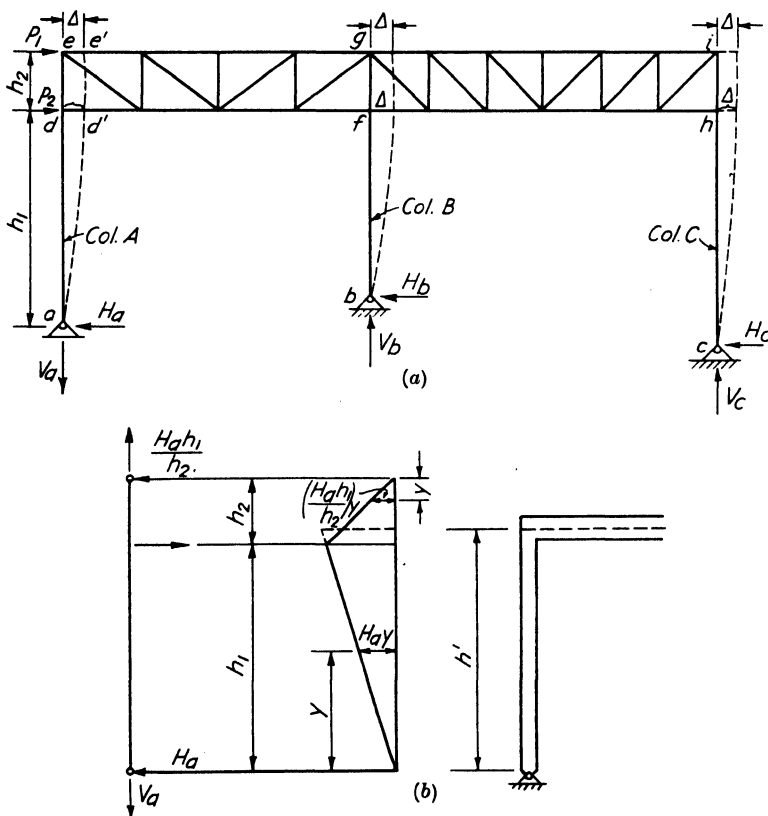



Fig. 8.15

and bending moments in the columns. This action in a bent, as such an integral structural unit of columns and trusses is called, is similar to the action in a continuous frame. In most bents, however, the trusses have much greater flexural rigidity as compared to the columns than do the girders in a frame. Moreover, as shown in Fig. 8.15*b*, the arm of the end couple acting between the truss and column is usually larger for the truss than for the girder, and, as the forces are more concentrated, the bending-moment diagram for a column in a bent has the shape shown by the solid line, whereas the diagram for a column in a frame, as usually assumed, is shown by the dotted line. Actually, the difference is probably less than that represented by these diagrams, but for all practical purposes the distribution shown is sufficiently accurate.

In the solution of bents it is convenient to make use of an equivalent frame and therefore to replace the bending-moment diagram for the bent by the dotted line for a frame. This substitution should be made so that a shearing force H will produce the same strain energy in the column of the substitute frame as for the original bent. If the column has a constant cross section throughout its length, the strain energy because of flexure is kept constant if

$$\int_0^{h_1} \frac{(Hy)^2}{2EI}\,dy + \int_0^{h_2} \frac{[(Hh_1/h_2)y]^2}{2EI}\,dy = \int_0^{h'} \frac{(Hy)^2}{2EI}\,dy$$

from which

$$h'^3 = h_1^3\left(1 + \frac{h_2}{h_1}\right) \tag{8.25a}$$

By means of equation 8.25*a*, the strain energy in the bent (Fig. 8.15*a*) can be expressed in the form

$$U = \frac{H_a^2 h_a'^3}{6EI_a} + \frac{H_b^2 h_b'^3}{6EI_b} + \frac{H_c^2 h_c'^3}{6EI_c} + \sum \frac{S^2L}{2AE} \tag{8.25b}$$

The strain energy in the columns is expressed in terms of the horizontal components H only, for the deformations produced by axial forces are relatively small, whereas the stresses S in the truss members are in terms of both H and V. From the equilibrium condition for the bent, the horizontal components must satisfy the equation

$$H_a + H_b + H_c = H \tag{8.26}$$

where H equals the total horizontal force.

8.10 Calculation of H Forces for Hinged Bents

The horizontal reactions for a bent that is hinged at the base of each column are readily determined from equation 8.25*b* by Castigliano's

theorem if the movement of each column base is known. By differentiating equation 8.25*b* with respect to H_a and taking

$$H_c = H - H_a - H_b$$

$$\frac{\partial H_c}{\partial H_a} = -1 \qquad \frac{\partial H_b}{\partial H_a} = 0 \tag{8.27a}$$

in which H_a and H_b are redundants and H_c is a reactive force, we obtain

$$\Delta_a = \frac{\partial U}{\partial H_a} = \frac{H_a h_a'^3}{3EI_a} - \frac{H_c h_c'^3}{3EI_c} + \sum \frac{S \dfrac{\partial S}{\partial H_a} L}{AE} = 0 \tag{8.27b}$$

In the same manner, by differentiating with respect to H_b, the expression for Δ_b becomes

$$\Delta_b = \frac{\partial U}{\partial H_b} = \frac{H_b h_b'^3}{3EI_b} - \frac{H_c h_c'^3}{3EI_c} + \sum \frac{S \dfrac{\partial S}{\partial H_b} L}{AE} = 0$$

For most bents, the numerical values of H_a, H_b, and H_c are affected but slightly by the terms

$$\sum \frac{S \dfrac{\partial S}{\partial H_a} L}{AE} \qquad \text{and} \qquad \sum \frac{S \dfrac{\partial S}{\partial H_b} L}{AE}$$

that is, by the deformation in the truss members. Consequently, by omitting these terms, equations 8.26, 8.27*a*, and 8.27*b* can be solved for the horizontal components, which can be expressed conveniently in the form

$$H_a = \left(\frac{C_a}{C_a + C_b + C_c}\right) H \tag{8.28a}$$

$$H_b = \left(\frac{C_b}{C_a + C_b + C_c}\right) H \tag{8.28b}$$

$$H_c = \left(\frac{C_c}{C_a + C_b + C_c}\right) H \tag{8.28c}$$

where the coefficient C is equal to

$$C = \frac{I}{h'^3}$$

and H is the total horizontal load on the bent. Obviously, these values of the horizontal reactions are the same as for a continuous frame in which the I/L value of the girders is equal to infinity and I/h' is used as the K value of the columns.

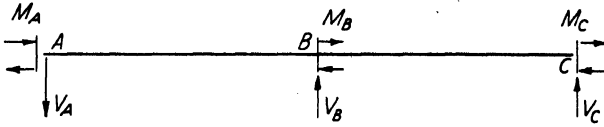



Fig. 8.16

8.11 *Calculation of Vertical Reactions*

After the horizontal reactions have been determined, the vertical components acting at the base of the columns must be calculated if the stresses in the truss members are desired. These vertical components are also statically indeterminate if there are more than two. To provide the necessary strain equations, the vertical displacements of the supports will be assumed equal to zero. Therefore, by differentiating equation 8.25*b* with respect to any one of the vertical components, say V_a, the equation for the vertical displacement of point *a* becomes

$$\Delta_v = \sum \frac{S(\partial S/\partial V_a)L}{AE} \qquad \text{(for all trusses)} = 0 \qquad (8.29)$$

Since the axial deformation in the columns has been neglected, the conditions represented by equation 8.29 are practically the same as for a continuous truss.

In Article 8.8 it was shown that a continuous beam can be selected that will approximate the behavior of a continuous truss and, for the problem under consideration, such a substitution is both convenient and justifiable from a design viewpoint. The problem therefore reduces to calculating the end couples and end shears in a continuous beam (Fig. 8.16) that is subjected to the effect of all external horizontal forces causing bent action. These horizontal forces can be transferred to the assumed axis of the equivalent beam by applying the necessary couple at that point. The location of the assumed axis of the equivalent beam cannot be determined mathematically, but, in general, it is natural and satisfactory to make it coincide with the axis of the truss. The numerical procedure will be illustrated by the following example.

Example 8.5 The horizontal and vertical components acting at the hinged bases of the columns in the bent shown (Fig. 8.17) will be determined for the uniform wind load of 0.6 kip per foot. Since the wind load is transmitted directly to column *A*, the numerical solution will be separated into two parts, that is,

(*a*) The local beam action in column *A* when points *a*, *d*, and *e* are assumed to have no horizontal displacement. This condition will require auxiliary restraining forces at points *d* and *e* as shown in Fig. 8.18*a*.

Fig. 8.17

(a)

(b)

(c)

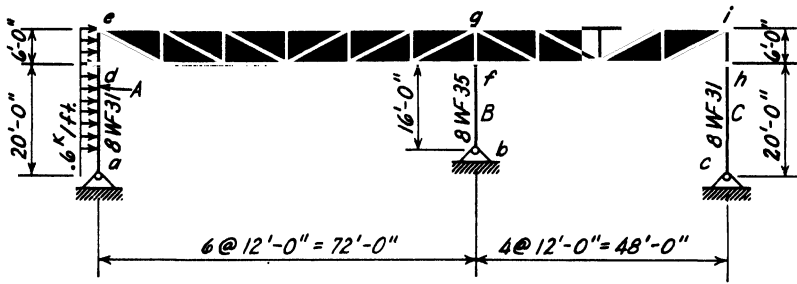
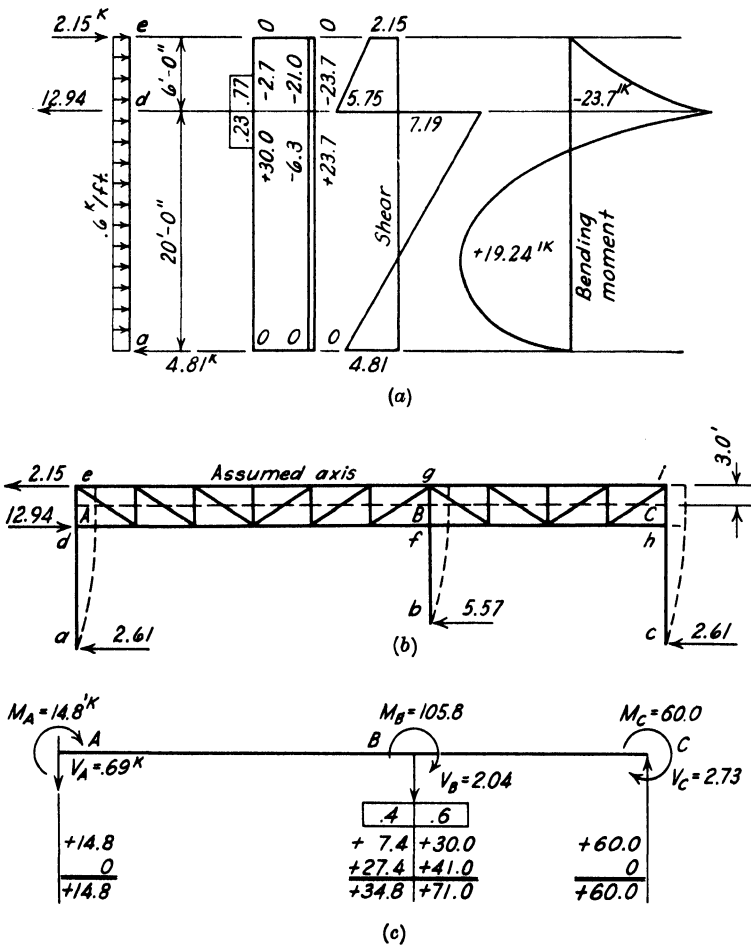



Fig. 8.18

(*b*) The removal of the auxiliary forces at d and e by applying equal and opposite forces as shown in Fig. 8.18*b*. This part of the solution is designated bent action as it involves horizontal displacement of the tops of all columns. Each column is assumed to be continuous from the base to the top of the truss.

The solution for part *a* by the moment-distribution method is given in Fig. 8.18*a*. Since the column is treated as a continuous beam of two spans only, one distribution of the unbalanced moments at support d is required. All shears and moments are shown in Fig. 8.18*a* as well as the auxiliary forces of 12.94 kips at d and minus 2.15 kips at e.

To remove these auxiliary forces, it will be necessary to apply a resultant force H to the right of magnitude.

$$H = 12.94 - 2.15 = 10.79 \text{ kips}$$

The shear coefficients C for the three columns are (removing $1/12^3$ from each)

$$\text{columns } A \text{ and } C \quad \frac{I}{(h')^3} = \frac{109.7}{20^3(1 + 6/20)} = 0.0105$$

$$\text{column } B \quad \frac{I}{(h')^3} = \frac{126.5}{16^3(1 + 6/16)} = 0.0225$$

$$C_A + C_B + C_C = 0.0435$$

From equation 8.28 the horizontal components at the base of each column due to the bent action is

$$H_a = H_c = \left(\frac{0.0105}{0.0435}\right) 10.79 = 2.61$$

$$H_b = \left(\frac{0.0225}{0.0435}\right) 10.79 = 5.57$$

Since one of the vertical components must be selected as a redundant, the procedure discussed in Article 8.11 will be used. It should be noted that the vertical components are due to the bent action only.

In this solution the axis of the equivalent continuous beam is assumed to be at the center of the truss or 3 ft below the top of all columns. The resultant couple M_A applied to the equivalent beam at end A (Fig. 8.18*c*) due to transferring the 2.61, 2.15, and 12.94 kip loads is

$$M_A = (2.61)(23) - (2.15)(3) - (12.94)(3) = +14.8$$

By transferring H_b to the axis at point B, we obtain

$$M_B = (5.57)(19) = +105.8$$

and, similarly for H_c,

$$M_C = (2.61)(23) = +60.0$$

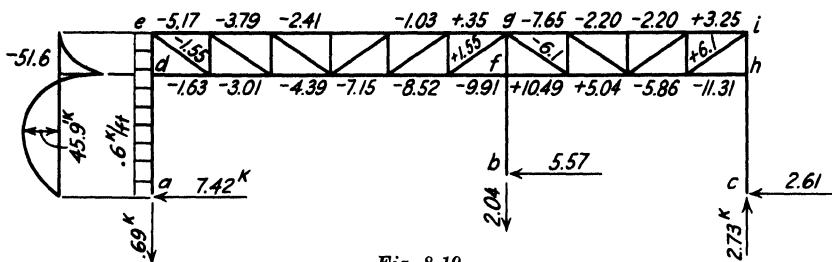



Fig. 8.19

The analysis of the equivalent continuous beam ABC subjected to these couples is shown in Fig. 8.18*c*. The distribution factors 0.4 and 0.6 were obtained by taking EI constant for both spans and by neglecting the stiffness of the columns relative to the trusses. Since the rotation at B was first assumed to be zero, one-half of the end couples M_A and M_C were carried over. Joint B was then balanced by the distribution factors 0.4 and 0.6. After the end moments are known, the end shears can be calculated in the usual manner, thus

$$V_{AB} = \frac{49.6}{72} = 0.69 \downarrow$$

$$V_{CB} = \frac{131.0}{48} = 2.73 \uparrow$$

Therefore the vertical reactions at the base of the columns are

$$V_a = 0.69\downarrow, \qquad V_b = 2.04\downarrow \qquad V_c = 2.73\uparrow$$

When the reactions due to both parts *a* and *b* are combined, the results shown in Fig. 8.19 are obtained. The stresses in the truss members indicate that only a few members connected to the columns are appreciably affected by bent action.

8.12 Bent Columns as Continuous Beams

In the preceding analysis it is apparent that the columns are considered as continuous members throughout their length, and therefore the same analytical methods as for continuous beams can be used. The moment-distribution method provides an ideal tool for general use as it is easily adapted to various base conditions, loading arrangements, and variations in cross section. If the trusses are assumed to be rigid bodies when investigating the flexural stresses in the columns due to bent action, only one horizontal displacement Δ is usually involved. However, other strain conditions can be readily incorporated into the solution if desired. The

application of the moment-distribution method to bent analysis is illustrated in the following example.

Example 8.6 The bent shown in Fig. 8.17 will also be used for this example except that columns *A* and *C* will be considered fixed at the base instead of hinged. As in Example 8.5, the solution is separated into the beam action and the bent action. The determination of the forces due to the beam action when auxiliary restraining forces are applied to prevent horizontal displacement at points *d* and *e* is shown in Fig. 8.20*a*. For a fixed-end condition at *a* the auxiliary forces at *d* and *e* are 9.92 and

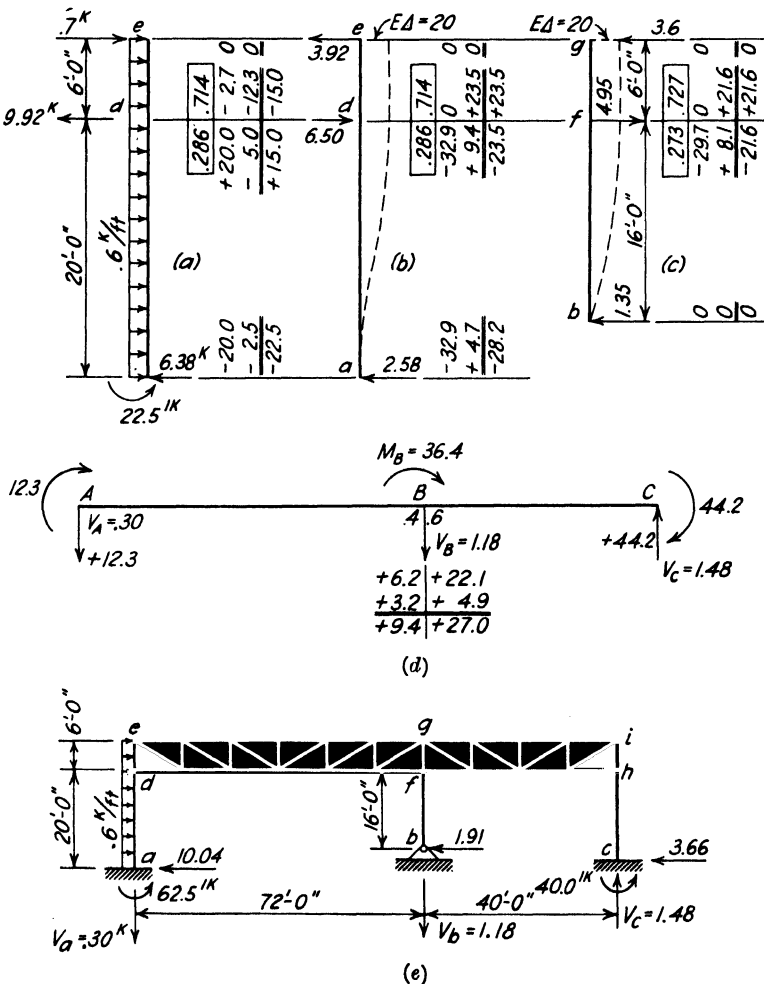



Fig. 8.20

minus 0.7 kips respectively and therefore a resultant horizontal force of 9.22 must be resisted by bent action.

To determine the forces acting on the columns due to bent action, the trusses will be assumed to have a horizontal displacement $E\Delta$ of 20 units to the right. For this displacement of all columns the following relative fixed-end moments will be obtained ($1/12^2$ omitted)

$$M_{ad} = M_{da} = M_{ch} = M_{hc} = -\frac{(6)(109.7)(20)}{(20)(20)} = -32.9$$

$$M_{fb} = -\frac{(3)(126.5)(20)}{(16)(16)} = -29.7$$

The distribution of the unbalanced moments at joint d for column A and h for column C is shown in Fig. 8.20*b* and in Fig. 8.20*c* for column B. The sum of the horizontal components at the base of all columns due to the assumed displacement of 20 units is 2.58 + 2.58 + 1.35, or 6.51 kips. Consequently, for the actual horizontal force of 9.22 kips the base forces and couples must be

$$H_a = H_c = \left(\frac{9.22}{6.51}\right)(2.58) = (1.41)(2.58) = 3.66 \text{ kips}$$

$$M_a = M_c = (1.41)(-28.2) = -40.0 \text{ t-kips}$$

$$H_b = (1.41)(1.35) = 1.91 \text{ kips}$$

To determine the vertical components, all external forces, including the couples, are applied to an equivalent continuous beam ABC Fig. 8.20*d* placed at the center of truss as in Example 8.5. The couples acting upon the equivalent beam are

$$M_A = (3.66)(23) - 40.0 - (9.92 + 0.7)(3) = +12.3$$

$$M_B = (1.91)(19) = +36.4$$

$$M_C = (3.66)(23) - 40.0 = +44.2$$

The same distribution factors of 0.4 and 0.6 at joint B as in Example 8.5 will be used. After the moments at joint B have been balanced and the shear in spans AB and BC calculated, the following column reactions were obtained.

$$V_a = 0.30\downarrow \qquad V_b = 1.18\downarrow \qquad V_c = 1.48\uparrow$$

The combined effects of both beam and bent action are given in Fig. 8.20*e*. When these results are compared to the corresponding values in Fig. 8.19 for hinged ends, it is apparent that the degree of restraint at the base of each column, which can only be estimated, exerts sufficient influence upon the structural behavior to justify the approximate methods that have been used.

8.13 Bents with Stepped Columns

Many mill buildings are constructed so that both crane girders and roof trusses can be supported directly on the same columns (Fig. 8.21)—an arrangement that usually necessitates a considerable change in the column section at the elevation of the crane girder. The preceding solution for mill bents in which the column is treated as a continuous beam is applicable to this problem if the proper column coefficients and fixed-end moments are used. Since the column section is usually constant between the base and the girder and also between the girder and the truss, the calculation of the distribution factors, carry-over factors, and fixed-end moments is not difficult. Moreover, the use of diagrams such as those on page 432 will greatly reduce the time required for the analysis. Since the method of solution has already been developed for bents with columns of constant section throughout, a numerical example will be sufficient to illustrate the procedure for columns with variable cross section.

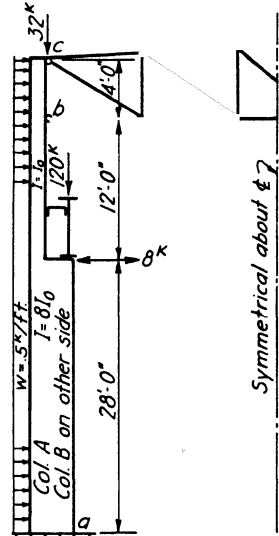



Fig. 8.21

Example 8.7[1] The maximum moments in columns *A* and *B* of the mill bent in Fig. 8.21 will be computed for various combinations of wind and crane loading by considering the columns as beams with variable moment of inertia. The wind load is taken as 0.5 kip per foot, and the crane loads as 120 kips vertical and 8 kips horizontal as shown. The horizontal crane load of 8 kips, which can be applied to either column to right or left, is assumed to be resisted by three bents because of the horizontal bracing.

For convenience in obtaining the maximum moments, the end moments for the three loads will be determined separately and then combined. For each load, the columns are first restrained against translation at *b* and *c*, by the auxiliary forces H_b and H_c, and then the necessary correction for sidesway is made. The base of each column is assumed fixed, and the ratio of I/I_0 is taken equal to 8. An assumption of approximately 85 to 90% of fixity at the base of the columns is probably more realistic. From

[1] This example was taken from a thesis, "Analysis of Mill Bents with Stepped Columns by the Method of Successive Approximations," by D. S. Ling, University of Michigan, 1943.

the curves on pages 432 and 433, the following coefficients are obtained for the member *ab*.

$$C_1 = 5.63 \qquad C_2 = 5.95 \qquad C_3 = 26.50$$

$$\text{Carry-over factor from } a \text{ to } b = \frac{5.95}{26.50} = 0.22$$

$$\text{Carry-over factor from } b \text{ to } a = \frac{5.95}{5.63} = 1.06$$

Distribution factors at b

Member	C	K	CK	r
ab	5.63	$\frac{I_0}{40}$	$0.141I_0$	0.16
bc	3.00	$\frac{I_0}{4}$	$0.750I_0$	0.84
			$\Sigma CK_0 = 0.891I_0$	1.00

The bending moments in the columns due to beam action for a wind load of 0.5 kip per foot will be calculated by the procedure explained in Example 8.5 and 8.6. The values for the fixed-end moments which are calculated directly from the curves on page 435 are

$$M_{Fab} = -0.111wL^2 = (-0.111)(0.5)(40)^2 = -88.8 \text{ ft-kips}$$
$$M_{Fba} = +0.0518wL^2 = 41.5 \text{ ft-kips}$$
$$M_{Fbc} = -0.125wL^2 = (-0.125)(0.5)(4)^2 = -1.0 \text{ ft-kip}$$

After the unbalanced moment at b, 40.5 ft-kips, is distributed by the distribution factors recorded previously, the final moments become (see Fig. 8.22*a*)

$$M_{ab} = -88.8 + (1.06)(0.16)(-40.5) = -95.7 \text{ ft-kips}$$
$$M_{ba} = -M_{bc} = 41.5 + (0.16)(-40.5) = 35.0 \text{ ft-kips}$$
$$H_a = 11.52 \text{ kips} \leftarrow \qquad H_b = 18.23 \text{ kips} \leftarrow \qquad H_c = 7.75 \text{ kips} \rightarrow$$

The end moments due to a horizontal crane load of 8 kips acting to the right can be quickly calculated once the fixed-end moments are determined. From the coefficients on page 434 the values of the fixed-end moments can be obtained directly.

$$M_{Fab} = -0.121PL = (-0.121)(8)(40) = -38.7 \text{ ft-kips}$$
$$M_{Fba} = 0.0994PL = +31.8 \text{ ft-kips}$$
$$M_{Fbc} = 0$$

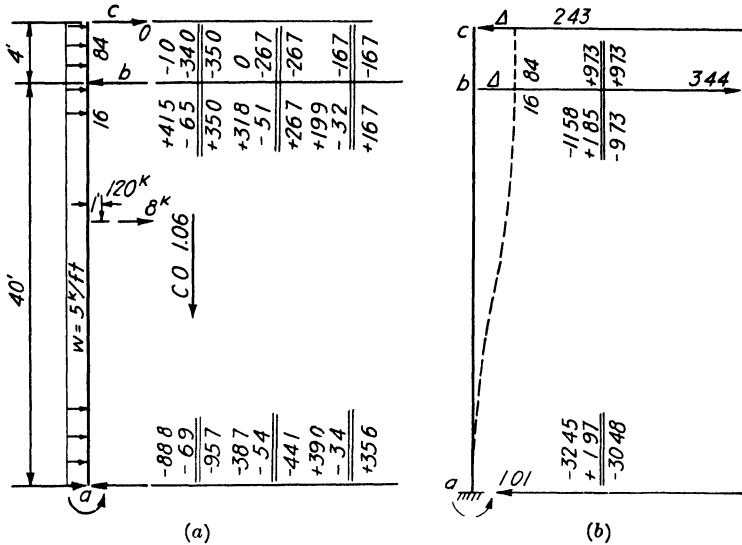



Fig. 8.22

After the necessary distribution of the unbalanced moment of 31.8 ft-kips at b, the final moments due to beam action are

$$M_{ab} = -38.7 - 5.4 = -44.1 \text{ ft-kips}$$

$$M_{ba} = -M_{bc} = 31.8 - 5.1 = 26.7 \text{ ft-kips}$$

$$H_a = 2.83 \text{ kips} \leftarrow \qquad H_b = 11.85 \text{ kips} \leftarrow \qquad H_c = 6.68 \text{ kips} \rightarrow$$

The fixed-end moments for the eccentric vertical load of 120 kips which is equivalent to an axial load of 120 kips and a couple of 120 ft-kips are also determined directly from the curves on page 436. These values are

$$M_{Fab} = 0.325M_x = +39.0 \text{ ft-kips}$$

$$M_{Fba} = 0.166M_x = +19.9 \text{ ft-kips}$$

which, after the distribution of the unbalanced moment at b (Fig. 8.22*a*) become

$$M_{ab} = 39.0 - 3.4 = 35.6 \text{ ft-kips}$$

$$M_{ba} = 19.9 - 3.2 = 16.7 \text{ ft-kips}$$

$$H_a = 4.31 \text{ kips} \rightarrow \qquad H_b = 8.49 \text{ kips} \leftarrow \qquad H_c = 4.18 \text{ kips} \rightarrow$$

The moments caused by the wind load and horizontal crane load will be corrected for sidesway by the customary procedure of applying forces

at points b and c that are equal and opposite to the restraining forces previously computed. Let points b and c both move 40 units to the right, and assume K_0 equal to unity. The relative values of the fixed-end moments for this condition are

$$M_{Fab} = (C_2 + C_3)K_0 \frac{\Delta}{L} = -(5.95 + 26.50)(1)\left(\frac{40}{40}\right) = -32.45 \text{ ft-kips}$$

$$M_{Fba} = (C_1 + C_2)K_0 \frac{\Delta}{L} = -(5.63 + 5.95)(1)\left(\frac{40}{40}\right) = -11.58 \text{ ft-kips}$$

The final moments obtained by rotating joint b to a condition of equilibrium (see Fig. 8.22*b*) are

$$M_{ab} = -32.45 + (1.06)(1.85) = -30.48 \text{ ft-kips}$$

$$M_{ba} = -11.58 + 1.85 = -9.73 \text{ ft-kips}$$

$$H_a = 1.01 \text{ kips} \leftarrow \qquad H_b = 3.44 \text{ kips} \rightarrow \qquad H_c = 2.43 \text{ kips} \leftarrow$$

The final moments in the columns for the separate loads can now be determined by adding the corrections due to sidesway to the original values. Thus, for the wind load, the correction for sidesway is obtained by multiplying the moments in Fig. 8.22*b* by the ratio 5.24/1.01 = 5.18, as the auxiliary force was 10.48 kips. Therefore the final values of the column moments due to the wind force are

Column A

$$M_{ab} = -95.7 + (5.18)(-30.48) = 254.2 \text{ ft-kips}$$

$$M_{ba} = +35.0 + (5.18)(-9.73) = -15.5 \text{ ft-kips}$$

$$H_a' = 11.52 + 5.24 = 16.76 \text{ kips}$$

Column B

$$M_{a'b'} = -158.5 \text{ ft-kips}$$

$$M_{b'a'} = 50.5 \text{ ft-kips}$$

$$H_{a'} = 5.24 \text{ kips}$$

The final moments in the columns resulting from the horizontal crane load are determined by correcting the original values for an additional shear of (8 − 2.83)/6, or 0.86 kip, acting on each column. The total restraining force of 5.17 kips is divided between six columns rather than two, for it is assumed that the horizontal bracing will distribute the load over three bents. The corrections of the end moments for sidesway are therefore obtained by multiplying the moments in Fig. 8.22*b* by 0.86/1.01,

or 0.852. The final moments in the columns resulting from the horizontal crane load are

Column A

$$M_{ab} = -44.1 + (0.852)(-30.48) = -70.1 \text{ ft-kips}$$
$$M_{ba} = 26.7 + (0.852)(-9.73) = 18.4 \text{ ft-kips}$$
$$H_a = 2.83 + 0.86 = 3.69 \text{ kips} \leftarrow$$

Column B

$$M_{a'b'} = -26.0 \text{ ft-kips}$$
$$M_{b'a'} = -8.3 \text{ ft-kips}$$
$$H_{a'} = 0.86 \text{ kip} \leftarrow$$

The moments resulting from the application of the vertical crane loads can also be corrected for sidesway if necessary. In this problem, since the vertical crane loads are assumed to be symmetrical, no correction for sidesway is required.

Maximum combined moments, assuming that the horizontal crane load can be applied to either column in either direction, are recorded below. These values which are presented to illustrate the numerical calculations are not intended to represent all possible loading conditions.

Column A

$$M_{ab} = \text{wind load} + \text{horizontal crane load} + \text{vertical crane load}$$
$$= -254.2 - 70.1 + 35.6 = -288.6 \text{ ft-kips}$$
$$M_{ba} = \text{horizontal crane load} + \text{vertical load} = 18.4 + 16.7 = 35.1 \text{ ft-kips}$$

Column B

$$M_{a'b'} = -158.1 - 70.1 - 35.6 = -263.7 \text{ ft-kips}$$
$$M_{b'a'} = \text{wind load} + \text{horizontal crane load} + \text{vertical crane load}$$
$$= -50.5 - 18.4 - 16.7 = -85.6 \text{ ft-kips}$$

8.14 *Two-Hinged Arch Trusses*

Two-hinged arch trusses in which the horizontal reactions are resisted either by the abutments or by a flexible tie (Figs. 8.23*a*, *b*) are readily

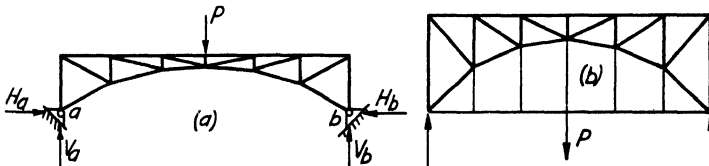



Fig. 8.23

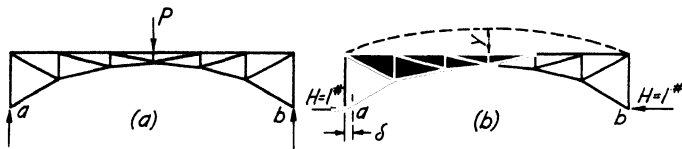



Fig. 8.24

solved by Castigliano's theorem. The stress S in any member is equal to

$$S = S' + Hu$$

in which S' equals stress in any member of a simply supported truss with H removed (Fig. 8.24*a*). u is the stress in any member of a simply supported truss for H equal to unity (Fig. 8.24*b*).

The total strain energy U equals

$$U = \sum \frac{S^2L}{2AE} = \sum \frac{(S' + Hu)^2L}{2AE}$$

Then

$$\frac{dU}{dH} = \sum \frac{(S' + Hu)uL}{AE} = 0$$

from which

$$H = -\frac{\sum \dfrac{S'uL}{AE}}{\sum \dfrac{u^2L}{AE}} \tag{8.30}$$

When the arch has a horizontal tie between supports, Fig. 8.23*b*, the denominator of equation 8.30 must include the quantity L/AE for the tie, but the numerator is not affected. The theoretical value of H will therefore be slightly less than when the arch is rigidly supported by abutments. Actually, a displacement of the abutments may reduce H more than the elongation of the tie bar. Since

$$\frac{dU}{dH} = \sum \frac{(S' + Hu)uL}{AE} = \Delta$$

therefore, for an increase in span, that is, a minus Δ,

$$H = -\frac{\sum \dfrac{S'uL}{AE} + \Delta}{\sum \dfrac{u^2L}{AE}} \tag{8.31}$$

It is interesting to note that the elongation of the tie bar increases the denominator in the expression for H, equation 8.30, whereas a displacement of the supports decreases the numerator in equation 8.31, for the summation term is negative. However, the two structures are not easily compared as the two displacements are likely to be quite different quantitatively. The use of a tie bar will usually depend on foundation conditions as well as other design features.

By drawing a Williot diagram for the arch for any assumed value of H, Fig. 8.24*b*, an influence diagram for H is obtained directly. From the reciprocal theorem,

$$H\delta - Py = 0$$

or

$$H = P\frac{y}{\delta}$$

therefore only y and δ, which are given directly by a Williot diagram, are required. This method of solution reduces the problem to a statically determinate one.

For preliminary designs a trial value of H can be obtained by assuming L/A as constant for all members. The trial value of H then becomes

$$H = -\frac{\sum S'u}{\sum u^2}$$

The final analysis with the proper areas should require only small corrections (see Ref. 1). Temperature changes will cause stresses in two-hinged arch trusses that are supported on abutments that prevent any horizontal movement. The horizontal reaction H due to any temperature change is equal to

$$H = \pm \frac{\alpha t L}{\sum \frac{u^2 L}{AE}} \tag{8.33}$$

where α = coefficient of linear expansion
t = change in temperature
L = span length

A tied arch is not affected by temperature change unless the tie bar is subjected to a different temperature from the truss.

8.15 Continuous Arch Trusses

The structure illustrated in Fig. 8.25, which is a combination of a continuous truss and a tied arch truss, is readily analyzed by the same

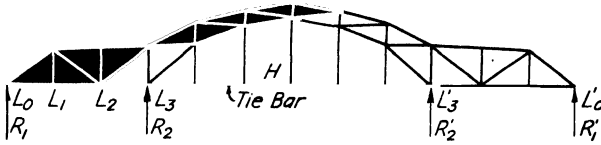



Fig. 8.25

methods as for continuous trusses (see Refs. 6 and 7). The redundant reactions R_1, R_2, H (or, if the structure is symmetrical about the center line, as it usually is, R_2, R_2', and H) can be calculated from equations 8.8*a*, *b*, and *c* by replacing one of the vertical reactions by H, the stress in the tie bar. If the structure must be designed for moving concentrated loads, the construction of influence diagrams for the redundant forces by the Müller-Breslau principle used in Example 8.1 is advisable.

The influence diagram for H is obtained by applying any convenient stress to the tie bar, say 10 kips (Fig. 8.26), calculating the redundant reactions R_2 and R_2', and then drawing a Williot displacement diagram for the structure. For a symmetrical structure R_2' is equal to R_2, and therefore ($E = 1$)

$$R_2 = -H \frac{\sum \frac{uu_2L}{A}}{\sum \frac{u_2{}^2L}{A} + \sum \frac{u_2'u_2L}{A}} \tag{8.34}$$

where u = stress in any member for H equal to unity with R_2 and R_2' removed

u_2 = stress in any member for R_2 equal to unity with H and R_2' removed

u_2' = stress in any member for R_2' equal to unity with H and R_2 removed

The values of u_2' are antisymmetrical with u_2 for a symmetrical structure.

The stress S in any member is equal to

$$S = Hu + R_2u_2 + R_2'u_2' = Hu + R_2(u_2 + u_2') \tag{8.35}$$

After the values of SL/A are tabulated, a Williot diagram is then drawn for half the structure. If the diagram is started at the pin L_3 with the

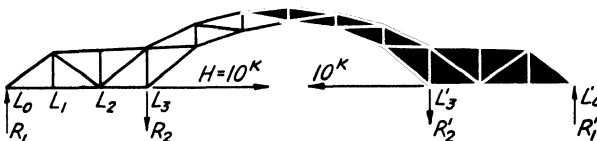



Fig. 8.26

chord L_2L_3 as the fixed axis, the construction errors are greatly reduced. A rotation diagram to bring point L_0 back to the required elevation must be constructed next, after which the necessary vertical and horizontal displacements can be scaled from the combined diagrams. The elongation of the tie bar must be added to the horizontal movement between the panel points L_3 and L_3' to obtain the total displacement. For a symmetrical structure the diagram need be drawn for only one-half of the truss.

The influence diagrams for R_2 and R_2' are constructed semigraphically in a similar manner by applying any convenient value of R_2, say 10 kips. The redundant forces R_2' and H must be determined next, after which the stress in each member is calculated and the Williot and Mohr displacement diagrams are drawn. The redundant forces R_2' and H are calculated from the following equations.

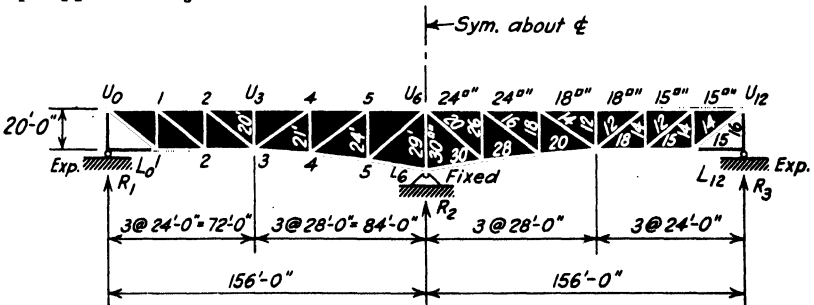
$$H \sum \frac{u^2L}{A} + R_2' \sum \frac{u_2'uL}{A} = -R_2 \sum \frac{u_2uL}{A} \tag{8.36a}$$

$$H \sum \frac{u_2'uL}{A} + R_2' \sum \frac{u_2'^2L}{A} = -R_2 \sum \frac{u_2u_2'L}{A} \tag{8.36b}$$

in which u, u_2u_2' have the same values as in equation 8.34. Although the displacement diagram must be drawn for the entire structure, the construction can proceed from one point or it can be made in two or more parts.

Problems

8.1 (*a*) Calculate the reactions for a total uniform load of 3 kips per linear foot of truss applied at the upper panel points. Use the dimensions shown on the left side of the diagram and the areas shown on the right side. Assume E to be constant.

(*b*) Calculate the reactions for a single concentrated vertical load of 30 kips applied at U_3.

Problem 8.1

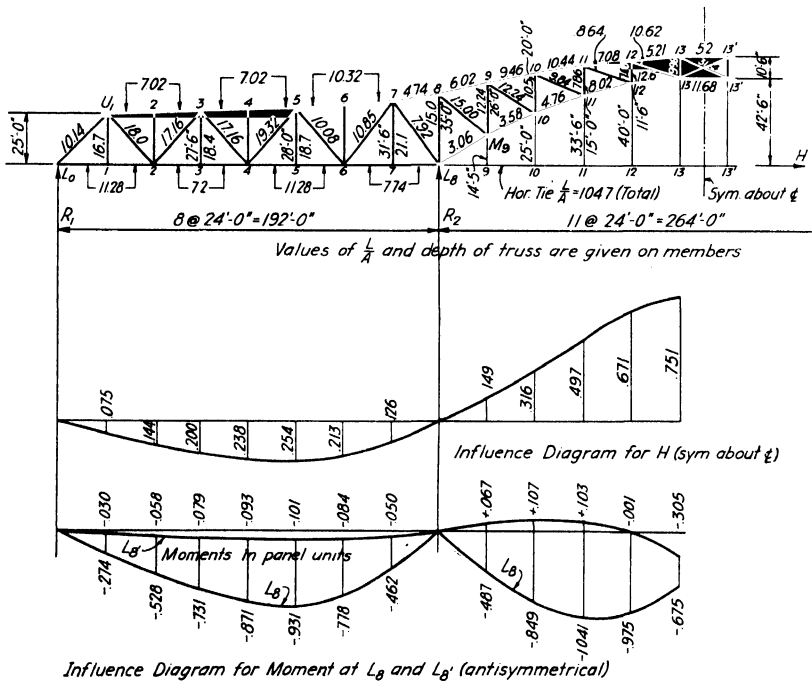
(*c*) Compare your answers in (*a*) and (*b*) with the corresponding values for a continuous beam of the same depth and with I varying as d^3. Use the diagrams in the Appendix for beams with parabolic haunches.

8.2 (*a*) Construct an influence diagram for the reaction R_2 in the truss of Problem 8.1 for a unit load applied at the upper panel points by the Müller-Breslau principle. Check the ordinate at U_3 with the results obtained from Problem 8.1*b*.

(*b*) From the results of part *a*, construct influence diagrams for the internal forces in members U_6L_6 and U_4U_5.

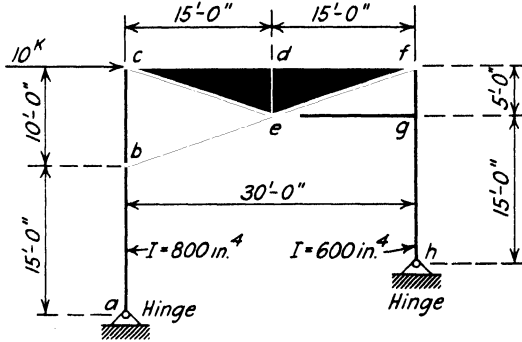
8.3 Construct an influence diagram for the reaction R_2 of the continuous truss in Example 8.1. Use $\sum \frac{u_1^2 L}{A} = 597.3$ and values of u_1 and u_2 as recorded in Figs. 8.5*a* and *b*.

8.4 A continuous arch-truss bridge that was built over the Meramec River by the Missouri State Highway Department (see Ref. 6) is shown in the diagrams. The areas of the various members are given in square inches.

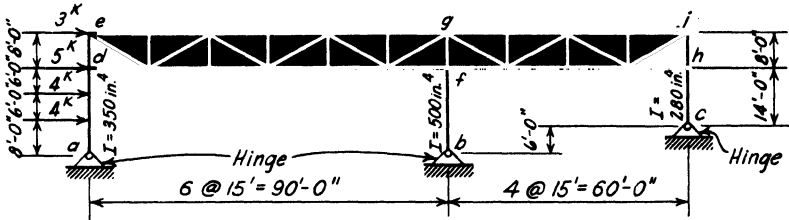
Construct an influence diagram for the force in the horizontal tie $L_8 - L_{8'}$ for a unit load applied at the lower panel points. Assume that the stresses in the center diagonals are of equal magnitude.

Problem 8.4

8.5 (*a*) Determine the reactions at the hinges *a* and *h* for the bent shown if the change in length of all members is neglected. Assume that members *abc* and *fgh* are continuous.

(*b*) Draw the shear and bending-moment diagrams for member *abc*.

(*c*) Check the horizontal reaction at *h* by Castigliano's theorem if the L/A value of all truss members is 30 in.$^{-1}$ and E is constant. Consider change in length of truss members but neglect axial strain in the columns.

Problem 8.5

8.6 Compare the horizontal displacements of point *c* of the bent in Problem 8.5 if the change in length of the truss members is both considered and neglected.

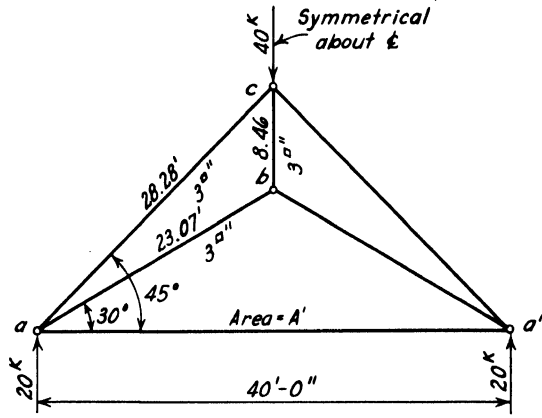
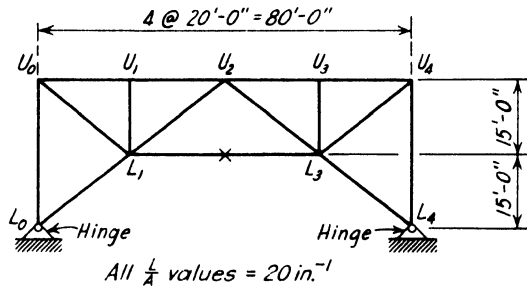
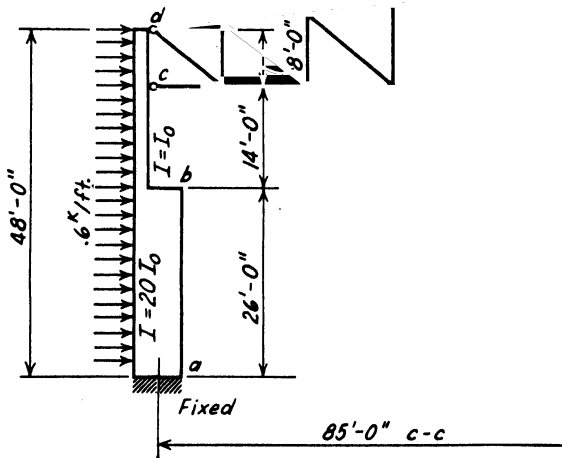
8.7 Determine the reactions at the base of each column of the bent shown. Neglect the deformation of the truss members. Draw the bending-moment diagram for member *ade* and give the value of the controlling ordinates.

Problem 8.7

8.8 Determine the reactions for the bent in Problem 8.7 if the columns are fixed at *a* and *b* and hinged at *c*. Consider the columns as continuous beams.

8.9 (*a*) What area A' is required for the tie aa' if the relative horizontal displacement between a and a' is 0.2 in.? Use E = 30,000 kips per square inch.

(*b*) What is the vertical displacement of point b?

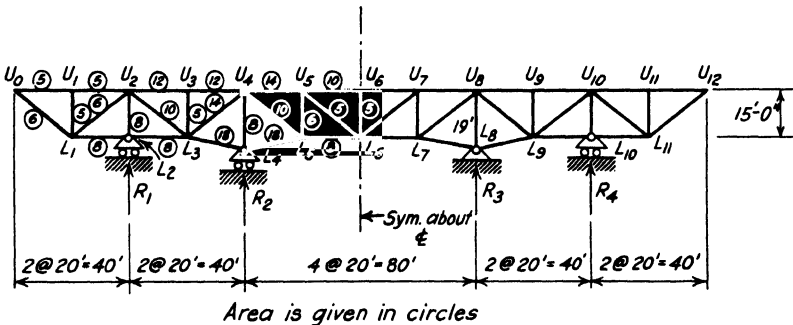
8.10 Construct an influence diagram for the horizontal component H for the two-hinged arch truss for a unit vertical load applied at the upper panel points. Use the indirect method that is based on the Müller-Breslau principle. All L/A values are constant as shown.

8.11 Calculate the stress in member L_1L_3 and the vertical displacement of U_2 for the arch truss in Problem 8.10 for a temperature rise of +60°F.

Problem 8.9

Problem 8.10

Problem 8.12

Use a coefficient of linear expansion of 6×10^{-6} per °F and $E = 30 \times 10^6$ lb per square inch.

8.12 Determine the shear and bending moments in the stepped column *abcd* of the bent shown for a wind load of 0.6 kips per foot. The bent has a similar column on the leeward side. Select the coefficients from the diagrams in the Appendix.

8.13 (*a*) Construct an influence diagram for R_2 by the indirect method using the Williot-Mohr vector diagrams for displacements. Consider a unit vertical load applied at the upper panel points. Give the value of the maximum ordinate in each span.

(*b*) Check the ordinates of this influence diagram at joints U_5 and U_7 by using the virtual work principle.

Problem 8.13 Cross-sectional areas of the members in square inches are circled. $E = 30 \times 10^6$ psi.

8.14 Construct an influence diagram for reaction R_1 of the continuous truss in Problem 8.13.

References

1 Albert Haertlein, "The Design of Statically Indeterminate Trusses," *J. Boston Soc. Civil Engrs.*, April 1936.

2 Jacobsen, "Moment Distribution and the Analysis of a Continuous Truss of Varying Depth," *Engg. Journal, Engg. Inst. of Canada*, December 1940.

3 P. C. Hu, "Comparison of Fixed-End Moments, Distribution Factors, and Carry-Over Factors in Trusses and Corresponding Beams," Thesis for M.S. degree in engineering, University of Michigan.

4 O. T. Voodhuigula, "Analysis of Statically Indeterminate Trussed Structures by Successive Approximations," *Trans. Am. Soc. C. E.*, Vol. 107 (1942).

5 D. S. Ling, "Analysis of Stepped-Column Mill Bents," *Trans. Am. Soc. C. E.* Vol. 113 (1948).

6 Howard H. Mullins, "Continuous Tied Arch Built in Missouri," *Engg. News-Record*, June 5, 1941, p. 896.

7 "A Three-Span Continuous Truss Bridge with the Middle Span a Tied Arch," *Engg. News-Record*, February 25, 1943, p. 42.

8 C. H. Gronquist, "Sault Ste. Marie International Bridge Design and Construction," *Civil Engineering*, May 1963.

9

Elastic Arches, Rings, and Frames with Curved Members

9.1 General Characteristics of an Arch

The structural advantages that result from using members with axes curved in a vertical plane have been known for many centuries; in fact, the intuitive recognition of the load-carrying capacity of a properly curved member probably accounts for its architectural popularity. As in many structural problems the transmission of forces through an arch should first be studied with respect to certain arbitrary boundary conditions and then the effect of the actual reactions, if different, can be determined. The reader should be aware, however, that the stresses in an arch can be substantially modified by changes in the boundary forces.

If the applied load on an arch is represented by a load intensity q per unit horizontal distance (Fig. 9.1a) and if the tangents to the axis of the arch at supports a and b are made to coincide with the resultant reactive forces R_a and R_b, the axis of the arch can theoretically be made to follow the funicular polygon due to the applied loads $q\,\Delta x$. Furthermore, the reactive forces R_a and R_b can be adjusted so as to pass the polygon and the assumed arch axis through any other point such as c. Since these operations should be familiar to the reader, they will not be discussed here. From a practical point of view such preliminary calculations as well as much of the final design can be made most conveniently on a drawing board although certain parts of the solution are most easily

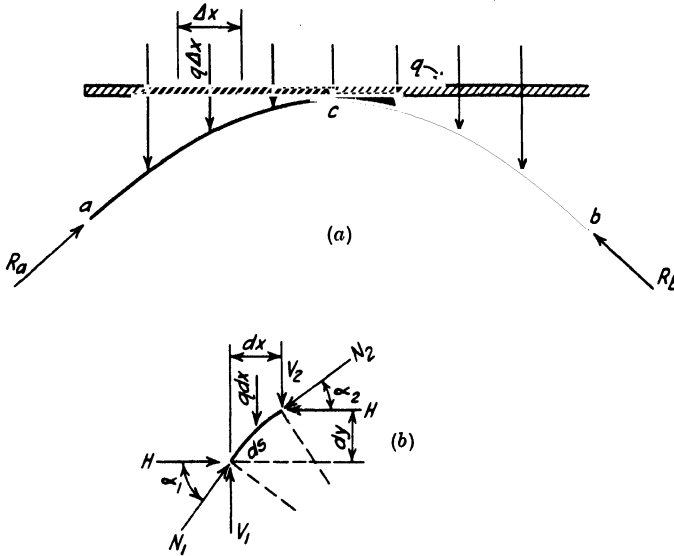



Fig. 9.1

performed by algebraic methods. The evaluation of the load q as well as recommendations for proportioning the arch rib will be discussed later.

Many algebraic and graphical procedures for determining the "best" location for the axis of an arch have been presented. Since a comprehensive discussion of this subject is presented in Ref. 11, only a few general comments will be made here. Most solutions of this problem utilize the equilibrium equation for the vertical forces acting on any element ds of the arch (Fig. 9.1b). Since the horizontal component H is constant,

$$V_1 - V_2 - q\,dx = 0$$

or

$$H(\tan \alpha_1 - \tan \alpha_2) = q\,dx$$

but, since

$$\tan \alpha_1 - \tan \alpha_2 = d\left(\frac{dy}{dx}\right)$$

then

$$\frac{d^2y}{dx^2} = \frac{q}{H} \tag{9.1}$$

All solutions developed from equation 9.1 will involve certain troublesome decisions. The following are examples.

1. What magnitude and distribution of q should be used? Should only the dead load be used or should some or all of the live load be included?

2. What adjustments should be made for the actual boundary conditions? If the arch is fixed at the abutments, should the axis be shifted to reduce the bending moments at the ends and center?
3. What effect will the volumetric changes caused by temperature variation and shrinkage have upon the final design?
4. Should any shifting of the arch axis due to movement of the abutments or the elastic displacements caused by the applied loads be considered? The latter problem will be treated in Chapter 10.

9.2 Design Procedures

In general, the design of most arches, especially in bridges, must be preceded by a study of such problems as clearance and functional requirements, foundation conditions, materials, construction details, live load and impact, allowable unit stresses, and other design criteria. If any important mistakes are made, it will probably be in the solutions to these problems.

The proportioning of the arch rib, although to a large degree dependent on the decisions made previously, will usually require the following steps:

1. Study of various approximate shapes of the arch axis as mentioned in Article 9.1. Graphical solutions are convenient for this part of the investigation.
2. Make some trial proportions at various sections and estimate values of the weights and moments of inertia. Refs. 1 and 11 will be found useful in this work.
3. When the live load is important, construct influence diagrams for the redundant reactions as well as for bending moments at several sections.
4. The effect of any temperature variation and the deformation in the piers or supporting members should be considered.

Analytical methods for performing items 3 and 4, as stated, will now be discussed in detail. However, the values obtained from these solutions should still be regarded as approximate because of the uncertainty of the strain conditions that must be assumed.

9.3 Two-Hinged Arches

Since an arch that is hinged at both supports has one redundant reaction (either horizontal component H), the analysis must begin by the determination of that force. Thus, in Fig. 9.2a, which shows a two-hinge solid rib or beam arch subjected to certain applied forces P_1, P_2, and P_3,

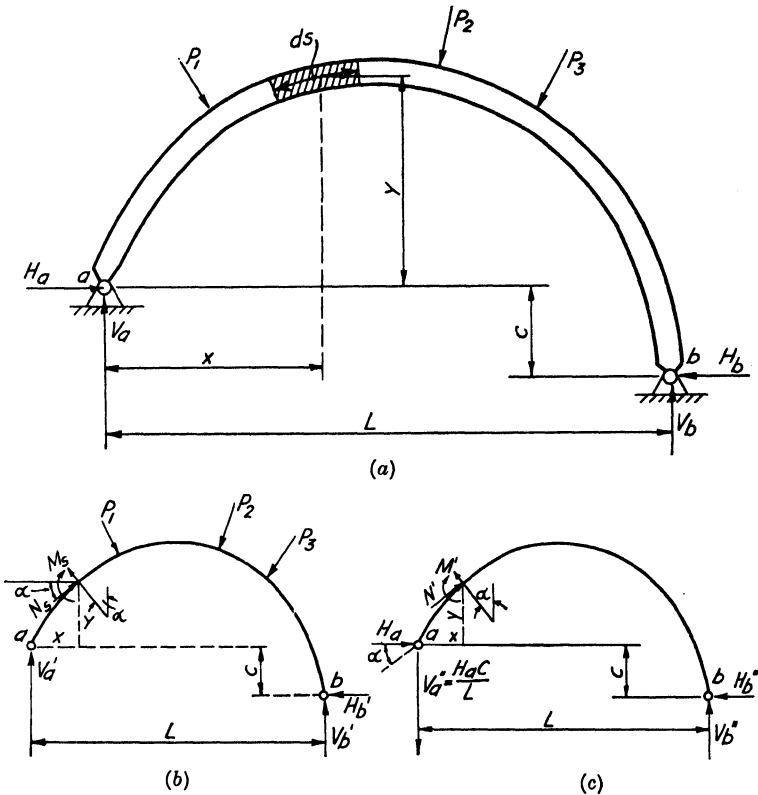



Fig. 9.2

the horizontal component H_a is taken as the unknown redundant force. The stresses and deformations in the arch can be expressed in terms of the two force systems shown in Figs. 9.2*b* and *c*, and for which the total strain energy (neglecting shear) is

$$U = \int_a^b \frac{(M_s + M')^2\,ds}{2EI} + \int_a^b \frac{(N_s + N')^2\,ds}{2AE} \tag{9.2}$$

The relative horizontal displacement between supports *a* and *b* is

$$\frac{\partial U}{\partial H_a} = \Delta_a$$

which, by equation 9.2, can be expressed in the form

$$\Delta_a = \int_a^b \frac{(M_s + M')(\partial M'/\partial H_a)\,ds}{EI} + \int_a^b \frac{(N_s + N')(\partial N'/\partial H_a)\,ds}{AE} \tag{9.3a}$$

But since

$$M' = -H_a\left(y + \frac{c}{L}x\right)$$

and

$$N' = H_a \cos\alpha - H_a \frac{c}{L} \sin\alpha$$

therefore

$$\frac{\partial M'}{\partial H_a} = -\left(y + \frac{c}{L}x\right)$$

$$\frac{\partial N'}{\partial H_a} = \cos\alpha - \frac{c}{L}\sin\alpha$$

If the values of $\partial M'/\partial H_a$ and $\partial N'/\partial H_a$ are substituted in equation 9.3*a* and the integration is replaced by summation, the expression for Δ_a becomes

$$\Delta_a = \frac{1}{E}\left[\sum_a^b - M_s\left(y + \frac{c}{L}x\right)\frac{\Delta s}{I} + \sum_a^b N_s\left(\cos\alpha - \frac{c}{L}\sin\alpha\right)\frac{\Delta s}{A}\right.$$
$$\left. + H_a \sum_a^b \left(y + \frac{c}{L}x\right)^2 \frac{\Delta s}{I} + H_a \sum_a^b \left(\cos\alpha - \frac{c}{L}\sin\alpha\right)^2 \frac{\Delta s}{A}\right] \quad (9.3b)$$

When Δ_a is equal to zero, the value of H_a is

$$H_a = \frac{\sum_a^b M_s\left(y + \frac{c}{L}x\right)\frac{\Delta s}{I} - \sum_a^b N_s\left(\cos\alpha - \frac{c}{L}\sin\alpha\right)\frac{\Delta s}{A}}{\sum_a^b \left(y + \frac{c}{L}x\right)^2 \frac{\Delta s}{I} + \sum_a^b \left(\cos\alpha - \frac{c}{L}\sin\alpha\right)^2 \frac{\Delta s}{A}} \quad (9.4a)$$

If the deformation due to the normal force is neglected and the supports are at the same elevation, that is, c is equal to zero, then equation 9.4*a* reduces to the simple expression

$$H_a = \frac{\sum_a^b \dfrac{M_s y \Delta s}{I}}{\sum_a^b \dfrac{y^2 \Delta s}{I}} \quad (9.4b)$$

Both numerator and denominator of equation 9.4*b* can be calculated for any applied loading on a given arch.

For arches with low rise/span ratios, that is, flat arches, the value of H is obtained more accurately by retaining the expression for the normal force in the denominator but omitting it in the numerator. If this is done, the calculated value of H, which is always less than when only the bending

moment is considered, is practically equal to the true value. For this assumption, the expression for H_a becomes

$$H_a = \frac{\sum_a^b \frac{M_s y \,\Delta s}{I}}{\sum_a^b \frac{y^2 \,\Delta s}{I} + \sum_a^b \cos^2 \alpha \frac{\Delta s}{A}} \tag{9.4c}$$

9.4 *Influence Diagram for H_a*

The value of the horizontal component H_a caused by a unit load applied to a two-hinged arch can be determined from equations 9.4*a*, *b*, and *c*. In these equations the denominator is constant for any particular arch since it does not involve the applied loads. The numerator, however, must be calculated for each position of the unit load, which requires considerable numerical calculations. In general, the indirect method provided by the Müller-Breslau principle, which utilizes the reciprocal theorem, has many advantages and therefore will be emphasized here.

The indirect method which transforms the unknowns from forces to displacements involves the use of an applied force H_a' (Fig. 9.3*b*) of any magnitude and the necessary reactive forces. This force system which is statically determinate causes the displacements shown by the dotted line. If the reciprocal theorem is now written for the two force systems shown in Figs. 9.3*a* and *b*, the following equation is obtained.

$$H_a \,\Delta_{ax} - P\delta = 0$$

which gives

$$H_a = P \frac{\delta}{\Delta_{ax}}$$

Consequently, the ordinates to the influence diagram for H_a in Fig. 9.3*a* are equal to the ratio δ/Δ_{ax} in Fig. 9.3*b*. This use of the reciprocal theorem should be clearly understood as it is frequently utilized for the construction of influence diagrams for redundant forces. The displacements in the

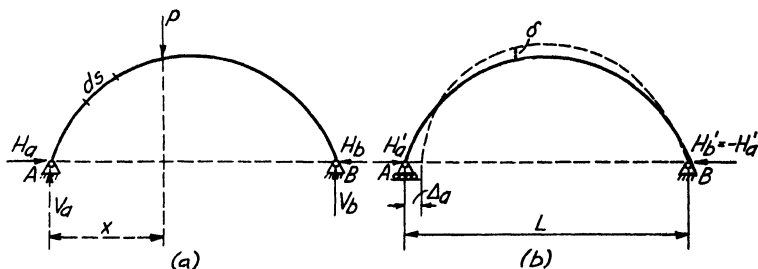



Fig. 9.3

direction of the load P are ordinarily calculated from the deformations in each element Δs due to bending moments only. However, as discussed later, for flat arches the effect of the normal forces should be included, particularly in determining the value of Δ_{ax}. Since the shortening of the arch axis, because of the compressive stress will increase the value of Δ_{ax}, the ordinates to the influence diagram for H_a will thereby be reduced. The procedure for calculating δ and Δ_{ax} by both numerical and graphical methods will be illustrated in the following examples.

Example 9.1 An influence diagram for H_a for the two-hinged circular arch of constant cross section (Fig. 9.4*a*) will be constructed by the

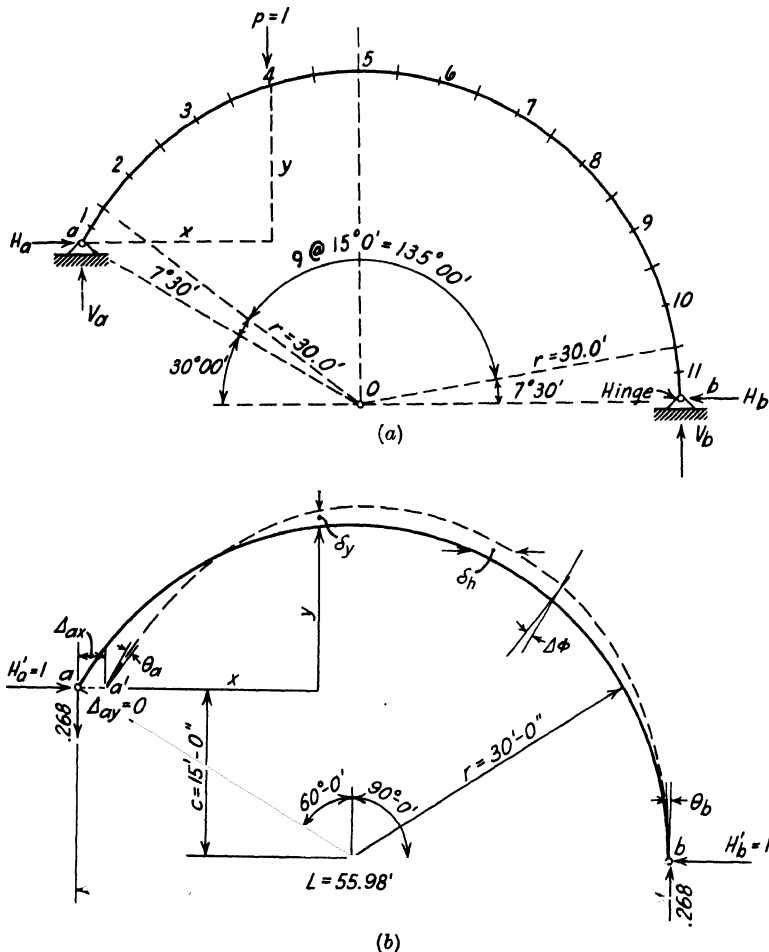



Fig. 9.4

indirect method discussed in Article 9.4. The auxiliary force system (Fig. 9.4*b*) consists of an applied load H_a' and a reaction H_b' of unity together with vertical reaction of c/L or 0.268. Any unit of force may be used.

The bending moment at any point on the arch axis whose coordinates are x and y with respect to point a (assuming counterclockwise moments positive for positive x and y)

$$M = (1)(y) + 0.268x$$

and the angle change $\Delta\phi$ for any element of length Δs is

$$\Delta\phi = \frac{M\,\Delta s}{EI} = GM$$

in which

$$G = \frac{\Delta s}{EI}$$

Since only relative values are required in the ratio δ/Δ_{ax}, the value of E, if constant, can be assumed unity and the values of $\Delta s/I$ may be divided by any constant. However, any consideration of absolute displacements will require absolute dimensional units. The relative values of $\Delta\phi$ for each element shown in Figs. 9.4*a* and *b* are recorded in Table 9.1.

Table 9.1

Point	x	y	$0.268x$	$G = \frac{\Delta s}{7.85}$	$\Delta\phi = G(y + 0.268x)$
a	0	0			
1	1.04	1.67	0.28	0.5	0.98
2	4.77	6.21	1.28	1	7.49
3	10.98	10.98	2.94	1	13.92
4	18.22	13.98	4.88	1	18.86
5	25.98	15.00	6.96	1	21.96
6	33.74	13.98	9.04	1	23.02
7	40.98	10.98	10.98	1	21.96
8	47.19	6.21	12.64	1	18.85
9	51.96	0	13.92	1	13.92
10	54.96	−7.24	14.73	1	7.49
11	55.92	−13.04	14.98	0.5	0.97
b	55.98	−15.00	15.00		

$$\sum_{1}^{11} \Delta\phi = 149.42$$

If the rotations of the end tangents are designated by θ_a and θ_b (Fig. 9.4*b*), their magnitudes can be calculated from the requirements that no vertical displacement exists at either *a* or *b*. Therefore since relative vertical motion can be calculated from the algebraic sum of the products of all angle changes times their corresponding horizontal distances, the vertical motion Δ_{ab} of point *a* with respect to *b* is equal to

$$\Delta_{ab} = 55.98\theta_b + \textstyle\sum_b^a x\Delta\phi = 0$$

or

$$\theta_b = \frac{-\sum_b^a x\Delta\phi}{55.98} = -\frac{4859}{55.98} = -86.80$$

As the $\Delta\phi$'s are counterclockwise in this equation, θ_b is clockwise. Similarly

$$\theta_a = \frac{-\sum_a^b (L - x)\,\Delta\phi}{55.98} = -62.62$$

In going from *a* to *b*, the $\Delta\phi$'s are clockwise and θ_a is counterclockwise and, as a check on the calculations

$$|\theta_a| + |\theta_b| = 149.42 = \textstyle\sum_a^b \Delta\phi$$

The values of δ_v can now be calculated from the products of angle changes times horizontal distance in the usual manner. Thus, beginning at point *b* where δ_v is zero, we obtain

$$\delta_{11} = (86.80)(0.06) = 5.21$$
$$\delta_{10} = 5.21 + (86.80 - 0.97)(0.96) = 87.61$$
$$\delta_9 = 87.61 + (85.83 - 7.49)(3.0) = 322.63$$

Table 9.2

Point	$\Delta\phi$	δ_v	y	$y\,\Delta\phi$	$H_a = \frac{(1)(\delta_v)}{2709.2}$
a		0			0
1	0.98	64.5	1.67	1.64	0.0238
2	7.49	294.4	6.21	46.51	0.1087
3	13.92	630.7	10.98	152.8	0.2328
4	18.86	922.6	13.98	263.7	0.3405
5	21.96	1088.4	15.00	329.4	0.4017
6	23.02	1083.8	13.98	321.8	0.4000
7	21.96	912.9	10.98	241.1	0.3369
8	18.85	629.9	6.21	117.1	0.2325
9	13.92	322.6	0	0	0.1191
10	7.49	87.6	−7.24	−54.2	0.0323
11	0.97	5.2	−13.04	−12.6	0.0019
b		0	$\sum_b^a y\,\Delta\phi$ =	1407.2	0

The reader should now check the other values of δ_v given in Table 9.2 by starting from point a. In a similar manner the value of Δ_{ax} can be determined from the equation

$$\Delta_{ax} = -15\theta_b + \textstyle\sum_b^a y\Delta\phi = (-15)(-86.80) + 1407.2 = 2709.2.$$

The numerical values of δ_v/Δ_{ax} which are the required ordinates to the influence diagram for H_a are recorded in Table 9.2. It should be noted that θ_b is separated from the $\Delta\phi$ values.

Example 9.2 The graphical construction of the influence diagram for H_a in Example 9.1 is given in Figs. 9.5*b* and *c* for the same angle changes and rotations of the end tangents as were used in the numerical solution. Each angle is constructed by the tangent method in accordance with the same assumptions that were used in the numerical solution, that is,

1. The difference of any two angles, such as $\theta_a - \Delta\phi_1$, is equal to $\tan(\theta_a - \Delta\phi_1)$. The error involved in this assumption has already been discussed in Article 3.4.
2. The intersection of the tangents occur at the center of the elements instead of the centroid of the $M\ \Delta s/EI$ diagram which is not practical to use.

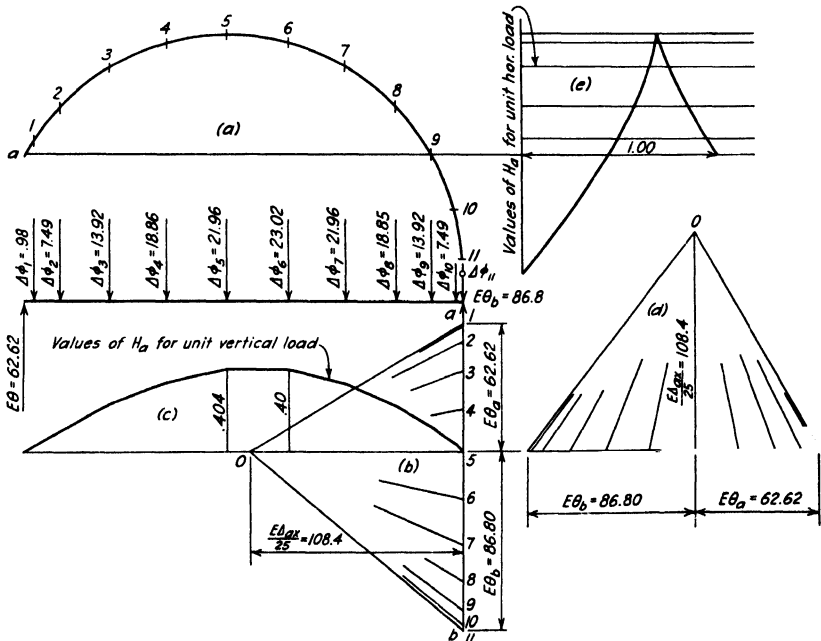



Fig. 9.5

3. If the slope of each tangent is divided by Δ_{ax}, the values of δ will be automatically divided by Δ_{ax}. This operation can be performed by making the pole distance in Fig. 9.5*b* equal to Δ_{ax}. However, if $\Delta_{ax}/2$ is used for the pole distance, the scale of the influence diagram is doubled. It should be noted that the scale of the influence diagram is always governed by the scale to which the arch axis is drawn and the pole distance used in obtaining the slope of the tangents.

In Fig. 9.5*b*, the actual slopes of the various tangents are constructed with a value of $\theta/(\Delta_{ax}/25)$ or the slope is $25\theta/\Delta_{ax}$. However, since 1 in. represents 10 ft of distance along the arch axis, the actual vertical displacement is $10\theta/\Delta_{ax}$ and therefore the vertical scale for H_a is 10/25, or 1 in. equals 0.4 lb. In general, the scale is m/n, where m is the scale to which the arch rib is drawn and the pole distance is Δ_{ax}/n as already explained. The numerical and graphical results check very well.

The values of H_a for a unit horizontal load acting at any point on the arch axis are equal to δ_h/Δ_{ax}, where δ_h is the horizontal displacement in Fig. 9.4*b*. To obtain δ_h, the slopes must be multiplied by the vertical distances, and therefore if the slopes are laid off with respect to the vertical axis, as in Fig. 9.5*d*, the influence diagram for H_a for a unit horizontal force can be constructed as shown in Fig. 9.5*e*. A check on the accuracy of the construction is obtained by starting at point *b* and proceeding toward *a*, at which the value of H_a should be unity.

9.5 *Fixed-End Arches*

The analysis of any arch that is restrained at both supports (Fig. 9.6) is usually started by resolving the external forces into known and unknown

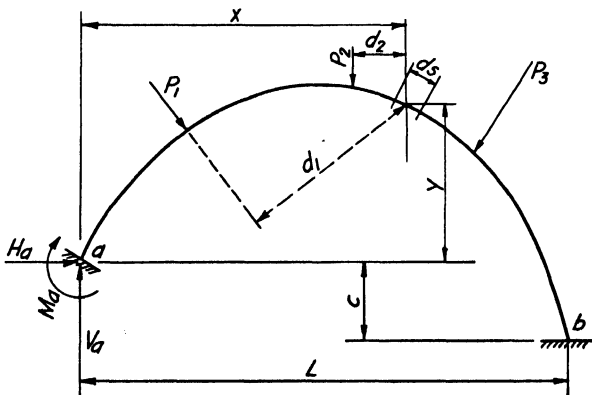



Fig. 9.6

force systems, the unknown forces, of course, being in terms of the redundant quantities M_a, V_a, and H_a.

The bending moment at any element ds is

$$M = M_a + V_a x - H_a y - P_1 d_1 - P_2 d_2 \tag{9.6}$$

and the total strain energy produced by the bending moments is

$$U = \int_a^b \frac{M^2\,ds}{2EI} = \int_a^b \frac{(M_a + V_a x - H_a y + M')^2\,ds}{2EI} \tag{9.7}$$

where M' equals $-P_1 d_1 - P_2 d_2$ equals moment of applied loads to the left of the element ds.

The linear and angular displacements of point a according to Castigliano's theorem are

$$\Delta_{ax} = \frac{\partial U}{\partial H_a} = \int_a^b \frac{(M_a + V_a x - H_a y + M')(-y)\,ds}{EI} \tag{9.8a}$$

$$\Delta_{av} = \frac{\partial U}{\partial V_a} = \int_a^b \frac{(M_a + V_a x - H_a y + M')(x)\,ds}{EI} \tag{9.8b}$$

$$\theta_a = \frac{\partial U}{\partial M_a} = \int_a^b \frac{(M_a + V_a x - H_a y + M')\,ds}{EI} \tag{9.8c}$$

By substituting G for $\Delta s/I$ and replacing the integration by summation, these equations can be written in the convenient form

$$E\,\Delta_{ax} = -M_a \sum Gy - V_a \sum Gxy + H_a \sum Gy^2 - \sum M'Gy \tag{9.9a}$$

$$E\,\Delta_{av} = M_a \sum Gx + V_a \sum Gx^2 - H_a \sum Gxy + \sum M'Gx \tag{9.9b}$$

$$E\theta_a = M_a \sum G + V_a \sum Gx - H_a \sum Gy + \sum M'G \tag{9.9c}$$

If the arch is fixed at a, then $\Delta_{ax} = \Delta_{av} = \theta_a = 0$, and equations 9.9*a*, *b*, and *c* can be solved for the redundant forces H_a, V_a, and M_a. This operation is laborious, but involves no particular difficulty.

Example 9.3 To illustrate the use of equations 9.9*a*, *b*, and *c* let us assume the elliptical arch shown in Fig. 9.7*a* to be fixed at both ends. In this arch the values $\Delta s/I = G$ are constant for each element and can be taken equal to unity. The value of H_a, V_a, and M_a will be calculated for a concentrated vertical load of 10 kips at the center of the span. The values of the various summations are

$$\sum x = 330 \qquad \sum y = 117.0 \qquad \sum xy = 3511$$

$$\sum x^2 = 14{,}059 \qquad \sum y^2 = 1435.3 \qquad \sum G = 11$$

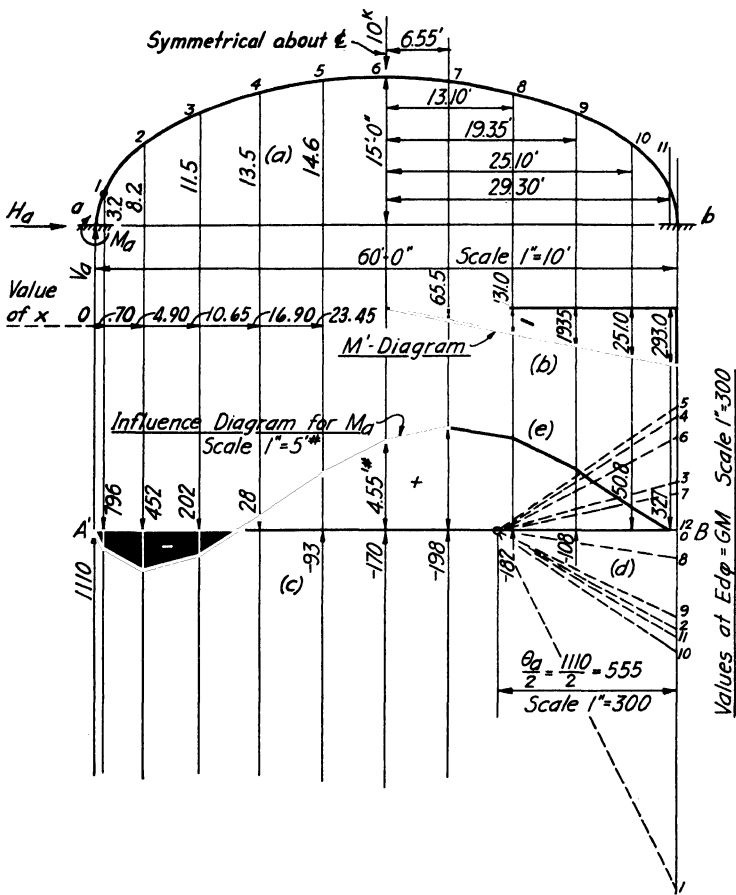



Fig. 9.7

The variation of M' is shown in Fig. 9.7*b*, and the values of the summations are

$$\sum M'G = -(65.5 + 131.0 + 193.5 + 251.0 + 293.0) = -934$$

Values of x = 36.55, 43.10, 49.35, 55.10, and 59.30.

$$\sum M'Gx = -(2394 + 5646 + 9549 + 13{,}830 + 17{,}375) = -48{,}794$$

Values of y = 14.6, 13.5, 11.5, 8.2, and 3.2.

$$\sum M'Gy = -(956 + 1769 + 2225 + 2058 + 938) = -7946$$

When these values are substituted in equations 9.9*a*, *b*, and *c*, we obtain

$$E\,\Delta_{ax} = -117M_a - 3511V_a + 1435H_a + 7946 = 0$$

$$E\,\Delta_{ay} = 330M_a + 14{,}059V_a - 3511H_a - 48{,}794 = 0$$

$$E\theta_a = 11M_a + 330V_a - 117H_a - 934 = 0$$

Solving, these equations give

$$H_a = 10.37 \text{ kips} \qquad V_a = 5.0 \text{ kips} \qquad M_a = 45.2 \text{ ft-kips}$$

9.6 Influence Diagram for M_a

If the force system shown in Fig. 9.8 is applied to the arch, the elastic curve can be used to obtain the influence diagram for M_a. By the reciprocal theorem

$$M_a\theta_a - P\delta = 0$$

or

$$M_a = P\frac{\delta}{\theta_a}$$

From equations 9.9*a* and *b*

$$E\,\Delta_{ax} = -M_a'\sum Gy - V_a'\sum Gxy + H_a'\sum Gy^2 = 0 \qquad (9.10a)$$

$$E\,\Delta_{ay} = M_a'\sum Gx + V_a'\sum Gx^2 - H_a'\sum Gxy = 0 \qquad (9.10b)$$

Equations 9.10*a* and *b* will give the value of V_a' and H_a' for any value of M_a'. The value of $E\theta_a$ can then be obtained from equation 9.9*c*, or

$$E\theta_a = M_a'\sum G + V_a'\sum Gx - H_a'\sum Gy \qquad (9.10c)$$

For any value of M_a' the bending moment M on an element Δs (Fig. 9.8) is

$$M = M_a' + V_a'x - H_a'y$$

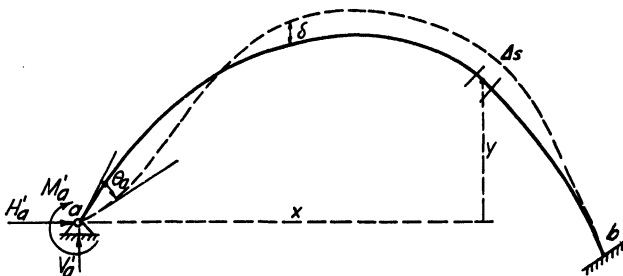



Fig. 9.8

and the rotation $\Delta\phi$ of each element Δs is

$$\Delta\phi = \frac{M\,\Delta s}{EI} \qquad \text{or} \qquad E\,\Delta\phi = GM$$

If each value of $E\,\Delta\phi$ or GM used as an elastic weight and $E\theta_a$ is used as a pole distance, the funicular polygon will be the influence diagram for M_a. The proof of this construction is practically the same as that in Example 9.2 for the two-hinged arch. However, here the tangent at b is fixed and therefore no correction for rotation is necessary.

Example 9.4 An influence diagram for the end moment of the elliptical arch used in Example 9.3 will be constructed by the semigraphical method.

Substituting the values of the summations given in Example 9.3 into equations 9.10a and b gives

$$-117M_a' - 3511V_a' + 1435H_a' = 0$$
$$330M_a' + 14{,}059V_a' - 3511H_a' = 0$$

from which

$$V_a' = -0.008M_a' \qquad H_a' = 0.062M_a'$$

and from equation 9.10c

$$E\theta_a = 11M_a' + (-0.008M_a')(330) - (0.062M_a')(117)$$

or

$$E\theta_a = 11M_a' - 2.64M_a' - 7.25M_a' = 1.11M_a'$$

If the value of M_a' is taken equal to 1000, then $V_a' = -8$, $H_a' = 62$, and $E\theta_a = 1110$. The value of M at each element is given in Table 9.3.

Table 9.3

Point	x	y	$-8x$	$-62y$	$M = 1000 - 8x - 62y$ $= E\,\Delta\phi$
1	0.70	3.2	−5.6	−198.4	796
2	4.9	8.2	−39.2	−508.4	452.4
3	10.65	11.5	−85.2	−713.0	201.8
4	16.90	13.5	−135.2	−837.0	27.8
5	23.45	14.6	−187.6	−905.2	−92.8
6	30.0	15.0	−240.0	−930.0	−170.0
7	36.55	14.6	−292.4	−905.2	−197.6
8	43.10	13.5	−344.8	−837.0	−181.8
9	49.35	11.5	−394.8	−713.0	−107.8
10	55.10	8.2	−440.8	−508.4	50.8
11	59.30	3.2	−474.4	−198.4	327.2

The value of $E\,\Delta\phi = GM = M$ is used as an elastic weight on the beam $A'B'$ (Fig. 9.7*c*) and a force polygon, Fig. 9.7*d*, is constructed for these forces beginning at point *b*. Since θ_b is zero, the tangent at point *b* is horizontal. A pole distance equal to

$$\frac{E\theta_a}{2} = \frac{1110}{2} = 555$$

has been adopted for the construction of the funicular polygon in Fig. 9.7, which is the influence diagram for M_a. The ordinates to this diagram are to twice the scale of the span of the beam $A'B'$, or 1 in. equals $\frac{10}{2}$, or 5 ft-lb.

9.7 *Influence Diagram for H_a*

An influence diagram for the horizontal reaction of a fixed-end arch can also be obtained by the indirect method. If the auxiliary force system shown in Fig. 9.9*a* is applied to the arch so as to give point *a* a horizontal displacement Δ_{ax} but with θ_a and Δ_{av} equal to zero, then, by equations 9.11*a* and 9.11*b*, the reactions V_a' and M_a' can be calculated for any value of H_a'.

$$E\,\Delta_{av} = M_a' \sum Gx + V_a' \sum Gx^2 - H_a' \sum Gxy = 0 \qquad (9.11a)$$

$$E\theta_a = M_a' \sum G + V_a' \sum Gx - H_a' \sum Gy = 0 \qquad (9.11b)$$

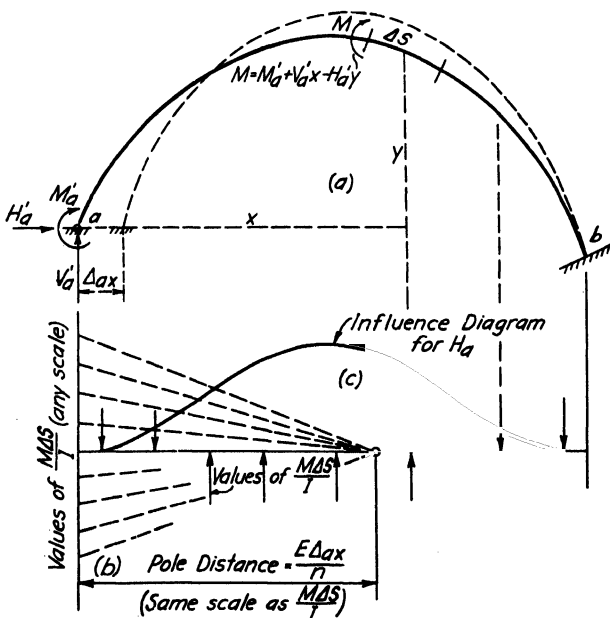



Fig. 9.9

The horizontal displacement $E\,\Delta_{ax}$ can now be calculated for any value of H_a' by equation 9.11*c*.

$$E\,\Delta_{ax} = -M_a' \sum Gy - V_a' \sum Gxy + H_a' \sum Gy^2 \qquad (9.11c)$$

The bending moment at any element of the arch in Fig. 9.9*a* is now known since

$$M = M_a' - H_a'y + V_a'x \qquad (9.12)$$

The vertical reaction V_a' will, of course, be zero for a symmetrical arch; otherwise the influence diagram for H_a would not be symmetrical.

The influence diagram for H_a is now drawn by laying off a force polygon (Fig. 9.9*b*) in which the $M\,\Delta s/I$ values are the elastic weights and the pole distance is equal to $E\,\Delta_{ax}/n$ (n is any convenient number). These quantities are drawn to any scale.

The funicular polygon (Fig. 9.9*c*) that is drawn from the force polygon (Fig. 9.9*b*) gives the influence diagram for H_a to the scale to which the span of the arch is drawn modified by the factor n.

Example 9.5 An influence diagram for the horizontal reaction H_a of the elliptical arch used in Examples 9.3 and 9.4 will be constructed by the indirect method just explained.

The values of the various summations for this arch are given in Example 9.3. If these values are substituted in equations 9.11*a* and *b*, we obtain

$$E\,\Delta_{av} = 330M_a' + 14{,}059V_a' - 3511H_a' = 0$$

$$E\theta_a = 11M_a' + 330V_a' - 117H_a' = 0$$

from which

$$M_a' = 10.64H_a' \qquad \text{and} \qquad V_a' = 0$$

When these values of M_a' and V_a' are substituted in equation 9.11*c*, the expression for $E\,\Delta_{ax}$ becomes

$$E\,\Delta_{ax} = -(10.64H_a')(117) + H_a'(1435.3) = 190.4H_a'$$

The bending moment at any element of the arch for H_a' equal to unity is

$$M = M_a' - H_a'y = 10.64H_a' - H_a'y = 10.64 - y$$

The value of G or $\Delta s/I$ is taken as unity for each element. The end points a and b have horizontal tangents, for the ends are held without rotation. In Fig. 9.10*a*, the values $M\,\Delta s/I$ are represented as elastic weights from which the force polygon is drawn with a scale of 1 in. equal to 5 units.

The pole distance is made equal to

$$\frac{E\Delta_{ax}}{10} = \frac{190.4}{10} = 19.04$$

and is drawn to the same scale. The funicular polygon constructed from the force polygon is the influence diagram for H_a. The scale of the arch span is 1 in. equals 10 ft, and therefore the scale of the influence diagram is 1 in. $= 10/n = 10/10 = 1$ lb. The ordinate at the center of the span scales about 1.05 lb, which practically checks the result of the algebraic

Fig. 9.10

Fig. 9.11

solution of Example 9.3. The vertical displacements can also be calculated numerically by the usual procedure of multiplying the angle changes times horizontal distance.

9.8 *Influence Diagram for* V_a

An influence diagram V_a can be drawn in the same manner as just described for M_a and H_a if the force system shown in Fig. 9.11*a* is applied to the arch. This force system must give some vertical displacement Δ_{ay} at point *a*, whereas Δ_{ax} and θ_a are kept equal to zero. The forces H_a' and M_a' that are necessary to give these displacements can be calculated from equations 9.13*a*, and *b* in terms V_a'.

$$E\,\Delta_{ax} = -M_a' \sum Gy - V_a' \sum Gxy + H_a' \sum Gy^2 = 0 \qquad (9.13a)$$

$$E\theta_a = M_a' \sum G + V_a' \sum Gx - H_a' \sum Gy = 0 \qquad (9.13b)$$

The vertical displacement $E\,\Delta_{ay}$ can then be calculated from the equation 9.13*c*.

$$E\,\Delta_{ay} = M_a' \sum Gx + V_a' \sum Gx^2 - H_a' \sum Gxy \qquad (9.13c)$$

The bending moment at any section, as for the preceding problems, is equal to

$$M = M_a' - H_a'y + V_a'x$$

For a symmetrical arch, H_a' is equal to zero and

$$M_a' = -\frac{V_a'L}{2}$$

where L is the span of the arch. Therefore, for a symmetrical arch,

$$M = V_a'\left(x - \frac{L}{2}\right)$$

Again the values of $M\,\Delta s/I$ (Fig. 9.11*b*) are used to lay off a force polygon from which the influence diagram for V_a is constructed. The pole distance

must be equal to $E\,\Delta_{ay}/n$, in which n is any convenient number. The force polygon can be constructed to any scale. The influence diagram for V_a will, of course, be the scale to which the arch span is drawn modified by the factor n.

Example 9.6 If the summations for the elliptical arch used in Examples 9.3, 9.4, and 9.5 are substituted in equations 9.13*a* and *b*, the expressions for $E\,\Delta_{ax}$ and $E\theta_a$ become

$$E\,\Delta_{ax} = -117M_a' - 3511V_a' + 1435.3H_a' = 0$$

$$E\theta_a = 11M_a' + 330V_a' - 117H_a' = 0$$

from which

$$H_a' = 0, \qquad M_a' = -30V_a' = -\frac{V_a'L}{2}$$

Therefore, if V_a' is taken equal to unity, $M_a' = -30$ and the bending moment M at any element is

$$M = x - 30$$

The values of $M\,\Delta s/I = M$ are represented as forces in Fig. 9.10*b*, and these values are used to draw the force polygon. The pole distance is taken equal to $E\,\Delta_{ay}/25$, which is found to be 166.7 by equation 9.13*c*. The influence diagram for V_a which is drawn from the rays of the force polygon is therefore to a scale of 1 in. $= \frac{10}{25} = 0.4$ lb. The ordinate at the center must be equal to 0.5 lb and at end a should be 1 lb.

9.9 *Effect of Temperature Change*

Any change in temperature produces a volumetric change which in an arch may cause stresses of considerable magnitude if the supports resist this motion. Thus, in any arch (Fig. 9.12) which is fixed at b and free to move at a, a rise in temperature will cause the end section mn to move to some position $m'n'$. This motion is expressed most conveniently in terms of the horizontal displacement Δ_{ax} and the vertical displacement

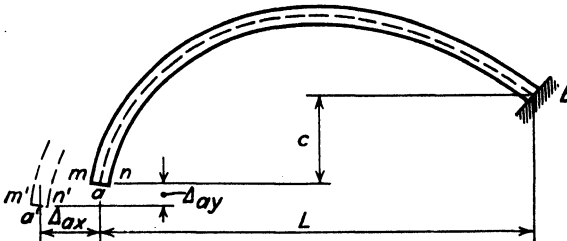



Fig. 9.12

Δ_{ay}, which are caused by the change in length of the horizontal and the vertical projections of the axis. Any rotation of the end section can be neglected since it is a secondary effect. These movements are easily calculated by the expressions

$$\Delta_{ax} = \alpha t L \tag{9.14a}$$

$$\Delta_{ay} = \alpha t c \tag{9.14b}$$

where α = coefficient of linear expansion
t = change in temperature
c = difference in elevation of supports a and b
L = horizontal projection of the axis between a and b

The values of H_a, M_a, and V_a necessary to prevent motion of the cross section at a can be calculated directly from equations 9.9a, b, and c. When the supports are at the same elevation, that is, with Δ_{ay} and θ_a equal to zero, the value of H_a is given directly by the solution of equations 9.11a, b, and c. These equations have already been solved in the construction of the influence diagram for H_a.

Example 9.7 The reactions and moment at the crown of the elliptical arch used in the preceding examples will be calculated for a rise in temperature of 50°F. The following data are used in the calculations:

$$t = 50^\circ \qquad \alpha = 6 \times 10^{-6} \qquad E = 2 \times 10^6 \text{ lb per square inch}$$

$$I = 0.667 \text{ ft}^4 \qquad G = \frac{\Delta s}{I} = \frac{6.8}{0.667} = 10.2 \text{ ft}^{-3}$$

$$E\,\Delta_{ax} = E\alpha t L = 5.17 \times 10^6 \text{ lb per foot}$$

From Example 9.5, the value of $E\,\Delta_{ax}$ is equal to

$$E\,\Delta_{ax} = 190.4 H_a G$$

when $\Delta_{ay} = \theta_a = 0$. Therefore

$$(190.4H_a)(10.2) = (5.17)(10^6)$$

$$H_a = 2660 \text{ lb}$$

$$M_a = 10.64H_a = 28{,}300 \text{ ft-lb}$$

$$M(\text{at crown}) = 28{,}300 - (2660)(15) = -11{,}700 \text{ ft-lb}$$

For a temperature drop, the signs of the forces would be reversed.

9.10 Relation between End Forces and End Displacements

In the preceding articles, equations have been developed for determining the relation between end forces and a given end displacement. Thus, by

means of equations 9.10a, b, and c, the end forces M_a', H_a', and V_a' can be expressed in terms of the end rotation $E\theta_a$, when $E\,\Delta_{ax}$ and $E\,\Delta_{ay}$ are equal to zero. In Example 9.4, these equations have the numerical values

$$E\theta_a = 1.11M_a'G \qquad \text{or} \qquad M_a' = \frac{0.9E\theta_a}{G}$$

$$H_a' = 0.062M_a' = \frac{0.0558E\theta_a}{G} \qquad V_a' = -0.008M_a'$$

By means of equation 9.11a, b, and c, the end forces M_a', H_a', and V_a' are expressed in terms of a given horizontal displacement $E\,\Delta_{ax}$ when $E\,\Delta_{ay}$ and $E\theta_a$ are set equal to zero. In Example 9.5, these equations gave numerical values of

$$E\,\Delta_{ax} = 190.4H_a'G \qquad \text{or} \qquad H_a' = \frac{E\,\Delta_{ax}}{190.4G}$$

and

$$M_a' = 10.64H_a' = 0.0558\frac{E\,\Delta_{ax}}{G}$$

From these relations the moment and horizontal components at the support can be calculated for a given horizontal displacement.

9.11 Stiffness and Carry-Over Factors for Curved Members

The term stiffness and carry-over factors have the same meaning for a curved beam as for a straight one. That is, if the end moment M_a' is equal to

$$M_a' = C_aE\theta_a \tag{9.15a}$$

C_a is the stiffness factor with respect to a rotation at a. The moment at b can be expressed in terms of M_a' or θ_a, that is,

$$M_{ba}' = C_{ab}M_{ab}' \tag{9.15b}$$

where C_{ab} is the carry-over factor. For example, we have already seen that for the elliptical arch used in preceding problems

$$M_a' = \frac{0.9E\theta_a}{G} \qquad \text{or} \qquad C_a = \frac{0.9}{G}$$

and $\quad M_b' = -M_a' + V_a'L = -M_a' + (0.008M_a')(60) = -0.52M_a'$.

Therefore the stiffness factor is $0.9/G$, and the carry-over factor is -0.52.

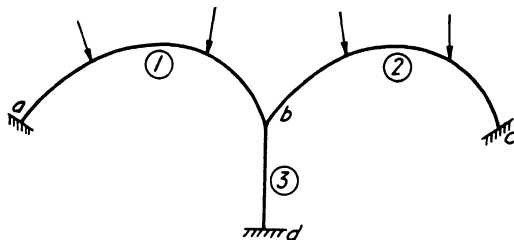



Fig. 9.13

9.12 Distribution Factors

After the stiffness coefficients have been calculated, the distribution factors are easily determined for any number of members, either straight or curved, that meet at a joint. For example, if the three members shown in Fig. 9.13 are rigidly connected at joint b, the end moments for all members can be expressed in terms of the fixed-end moments and the moments due to the rotation θ_b and horizontal displacement Δ_{bx}.

The difference between a frame composed of straight members and one with curved beams consists primarily in the larger horizontal force applied to the joint from the curved members and the possibility of joint b having a relative horizontal displacement with respect to a and c. For this reason there will usually be more unknown displacements to consider for frames with curved members than for frames with straight ones. It is always necessary to assume first that an auxiliary force or reaction is applied at the joint so as to prevent any horizontal movement. In other words, if the moment-distribution method is to be used, the joint must not be permitted to translate while it is given a rotation. If the members in Fig. 9.13 undergo only a rotation at joint b,

$$M_{ba} = M_{Fba} + C_{b1}E\theta_b = M_{Fba} + M_{ba}'$$

$$M_{bc} = M_{Fbc} + C_{b2}E\theta_b = M_{Fbc} + M_{bc}'$$

$$M_{bd} = C_{b3}E\theta_b = M_{bd}'$$

To satisfy the equilibrium condition at joint b,

$$M_{ba} + M_{bc} + M_{bd} = 0$$

or

$$M_{Fba} + C_{b1}E\theta_b + M_{Fbc} + C_{b2}E\theta_b + C_{b3}E\theta_b = 0$$

from which

$$E\theta_b = -\frac{M_{Fba} + M_{Fbc}}{C_{b1} + C_{b2} + C_{b3}} = -\frac{\sum M_{Fb}}{\sum C_b} \tag{9.16}$$

Therefore the corrections to the fixed-end moments are

$$M_{ba}' = \frac{C_{b1}}{\sum C_b}(-\sum M_{Fb}) \tag{9.17a}$$

$$M_{bc}' = \frac{C_{b2}}{\sum C_b}(-\sum M_{Fb}) \tag{9.17b}$$

$$M_{bd}' = \frac{C_{b3}}{\sum C_b}(-\sum M_{Fb}) \tag{9.17c}$$

The change in the horizontal components acting on the arches which are caused by the preceding change in end moments M' can be calculated from the relations given in Article 9.10.

Example 9.8 The reactions for the frame in Fig. 9.14*a* will be calculated for a uniform load of 1 kip per foot by the moment-distribution method. This method of analysis has no particular merit over other procedures for one-span frames or for one type of loading. When the problem involves several spans and various loading arrangements, however, the moment-distribution method, if thoroughly understood, is often the most advantageous.

The solution of the problem must begin with the determination of the reactions for the arch member for full restraint at points *a* and *b*. If the indirect method for the construction of influence diagrams is used for this part of the problem, the necessary data for calculating the stiffness and carry-over factors will also have been obtained. This sequence of numerical operations is illustrated in the present problem, for the member *ab* is identical with the elliptical arch for which the influence diagrams for M_a, H_a, and V_a were drawn in Examples 9.4, 9.5, and 9.6, respectively. From Example 9.4 we obtain not only the fixed-end moments for any type of vertical loads but also the stiffness and carry-over factors. The magnitude and direction of the fixed-end moments which are determined from the area of the influence diagram for M_a in Fig. 9.7 are

$$M_{Fab} = +126 \text{ ft-kips}$$

$$M_{Fba} = -126 \text{ ft-kips}$$

It should be noted that these signs are the reverse of those for a straight beam. In Example 9.4 the rotation θ_a for any moment M_a' applied at end *a* with end *b* fully restrained was found to be

$$E\theta_a = 1.11M_a'$$

Analysis of Single Span Arch Bent
w=1K/ft.
Ellipse
15'
I=constant
a
b
60'
20'
(a)
c
d
Fixed
Fixed
Summary of solutions from preceding examples
w=1K/ft.
(b)
36.0K
θa=0
126'K
30K
θb=0
126'K
36.0K
30K
(c)
θa
M'a
M'ab=.132IEθa
H'ab
H'ab=.062M'a
V'ab
M'ba=-.52M'ab
H'ba=-H'ab
θb=0
M"ab=.00822IEΔax
(d)
M"ab
H"ab
θa=0
H"ab=M"ab/10.64=.000772IEΔax
Δax
M"ba=-M"ab
H"ba=-H"ab
M'ac
M'ac=4EI/l θa=4EI/20 θa=.2IEθa
θa
20'
(e)
θc=0
M'ca=.5M'ac

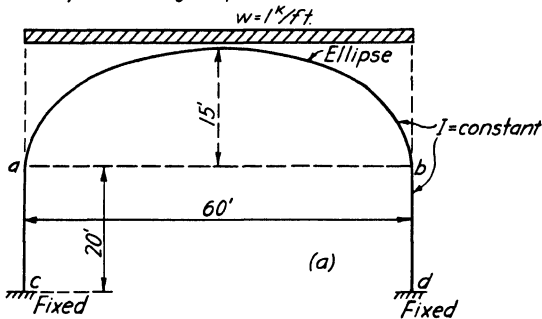
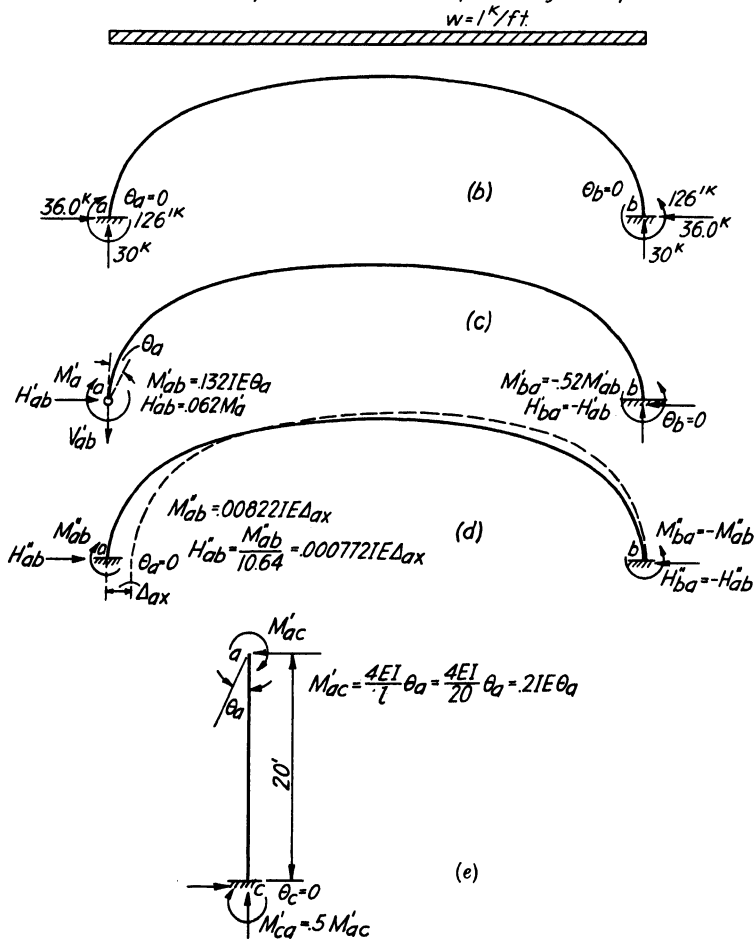



Fig. 9.14

However, in that solution the relative value of $\Delta s/I$ was taken equal to unity, whereas the actual value is approximately $6.8/I$ (scaled). Therefore

$$E\theta_a = 1.11\left(\frac{6.8}{I}\right)M_a'$$

or

$$M_a' = 0.132EI\theta_a$$

giving the stiffness factor as $0.132I$.

In the same solution, the corresponding value of H_a' and V_a' were found to be

$$H_a' = 0.062M_a'$$

$$V_a' = -0.008M_a'$$

The value of the fixed-end moment at b for a moment applied at a is therefore (taking summation of moments about b equal to zero)

$$M_a' + M_b' - (0.008M_a')(60) = 0$$

or

$$M_b' = -M_a' + 0.48M_a' = -0.52M_a'$$

Consequently, the carry-over factor is -0.52.

From Example 9.5, the value of the horizontal component H_a is obtained for any vertical load on the arch, and in addition the reactions are expressed in terms of any horizontal motion Δ_{ax}.

The horizontal component H_a for a uniform load of 1 kip per foot is found to be 36.0 kips from the area of the influence diagram that was obtained in Example 9.5. The relation between the end forces and the horizontal displacement Δ_{ax} was found to be

$$E\Delta_{ax} = 190.4H_a\frac{\Delta s}{I}$$

from which (Fig. 9.14*d*)

$$H_{ab}'' = \frac{EI\Delta_{ax}}{(190.4)(6.8)} = 0.000772E\Delta_{ax}I$$

The corresponding values of M_{ab}'' and V_{ab}'' are

$$M_{ab}'' = 10.64H_{ab}'' = 0.00822E\,\Delta_{ax}I$$

$$V_{ab}'' = 0$$

The values of V_a and the relation between the end forces and vertical displacement Δ_{ay} are given in Example 9.6. Since no vertical displacements are considered in the present problem, these relations are not needed.

The distribution factors at joints a and b for the arch ab and columns are (see Fig. 9.14*e*)

Member	CK	r	Carry-Over
ac	$0.2I$	0.6	0.5
ab	$0.132I$	0.4	−0.52
	$\Sigma CK = 0.332I$	1.00	

The distribution of the fixed-end moments, which is recorded in Fig. 9.15*a*, is similar to the procedure for bents with straight members except that the corrections M_{ab}' and M_{ba}' are recorded in a separate column from the fixed-end moments. This arrangement is necessary for the change in the horizontal component H_a produced by the rotation of the joints must also be calculated. The auxiliary forces F prevent any horizontal displacement of joints a and b.

From the values recorded in Fig. 9.15*a* the end moments and shears are equal to

$$M_{ca} = -31.3 \text{ ft-kips} \qquad M_{ac} = -62.6 \text{ ft-kips}$$

$$H_{ca} = \frac{31.3 + 62.6}{20} = 4.70 \text{ kips} \leftarrow$$

$$M_{ab} = 104.3 - 41.7 = 62.6 \text{ ft-kips}$$

$$H_{ab} = H_{Fab} + 0.062(M_{ab}' - M_{ba}')$$

or

$$H_{ab} = 36.0 + 0.062(-41.7 - 41.7) = 30.83 \text{ kips}$$

Therefore, for equilibrium at joints a and b,

$$F = 30.83 + 4.70 = 35.53 \text{ kips}$$

The two auxiliary forces F must now be removed by the same procedure as was followed in Chapter 6 for frames with straight members. By this method, joints a and b are given any assigned horizontal displacement (say $E\,\Delta_x = 1000$), whereas the rotation and vertical displacements are kept zero. Since we have already obtained the forces in terms of the displacement of joint a, these values can be multiplied by two to give

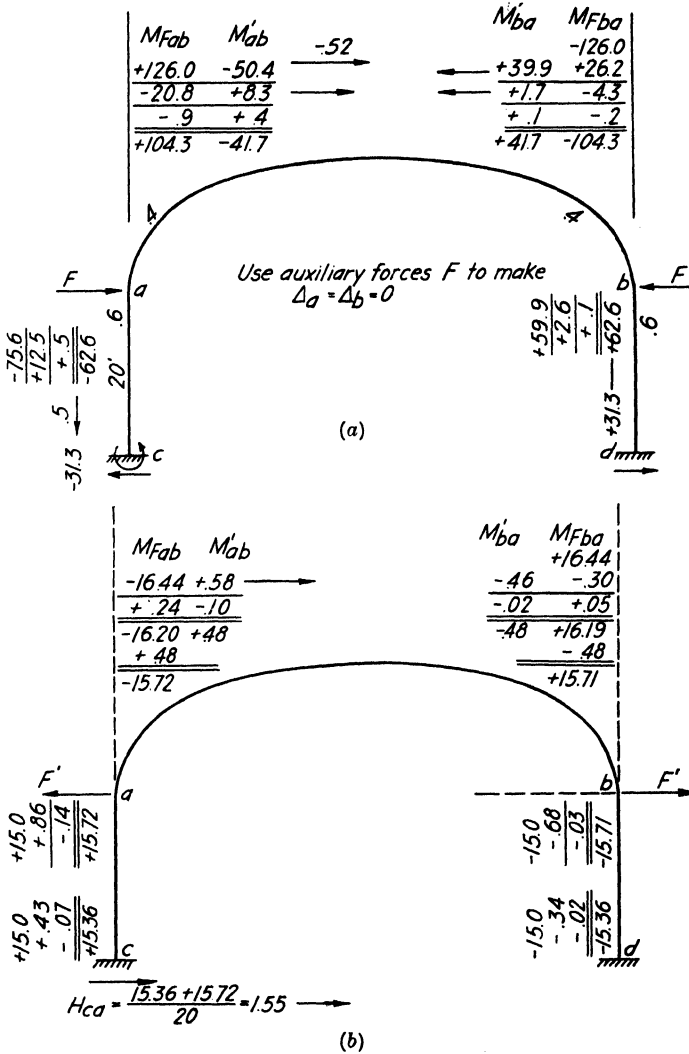



Fig. 9.15

the values for both displacements, or, if $\Delta_{ax} = -\Delta_{ax}$,

$$H_a = -(0.000772EI\Delta_{ax})2 = -1.544I \text{ kips}$$

$$M_{Fab} = -2(0.00822EI\Delta_{ax}) = -16.44I \text{ ft-kips}$$

$$M_{Fba} = +16.44I \text{ ft-kips}$$

$$M_{Fac} = M_{Fca} = \frac{6IE\Delta_{ax}}{(20)(20)} = \frac{6000I}{400} = 15I \text{ ft-kips}$$

The value of I is taken as unity because only the proper relative stiffness is required.

The distribution of these fixed-end moments is recorded in Fig. 9.15*b*. The procedure and arrangement are similar to those in Fig. 9.15*a*. The value of F' required to give the assigned displacement is therefore

$$F' = H_{ca} + H_{ab}$$

$$H_{ca} = \frac{15.36 + 15.72}{20} = 1.55 \text{ kips} \rightarrow$$

$$H_{ab} = -1.544 + 0.062(0.48 + 0.48) = -1.486 \text{ kips}$$

Therefore

$$F' = 3.036 \text{ kips}$$

To remove the auxiliary forces F in Fig. 9.15*a* the moments and shears in Fig. 9.15*b* must be multiplied by

$$\frac{35.53}{3.036} = 11.72$$

and added to those in Fig. 9.15*a*.

The final values are

$$M_{ca} = -31.3 + (11.72)(15.36) = 148.7 \text{ ft-kips}$$

$$M_{ab} = 62.6 - (11.72)(15.72) = -121.7 \text{ ft-kips}$$

$$H_c = \frac{148.7 + 121.7}{20} = 13.52 \text{ kips}$$

$$H_{ab} = 30.83 - (11.72)(1.486) = 13.42 \text{ kips}$$

$$M_{\text{crown}} = \frac{wL^2}{8} - 121.7 - (13.42)(15) = 127.0 \text{ ft-kips}$$

Example 9.9

ANALYSIS OF A SINGLE-SPAN ARCH BENT FOR A CONCENTRATED LOAD AT THE CENTER. The end moments acting on the members of the arch bent that was used in Example 9.8 will be calculated for a load of 10 kips (Fig. 9.16) applied at the center of the span.

The fixed-end forces acting on the arch *ab* are given by the influence diagrams in Figs. 9.7*e* and 9.10*a* and *b*. Therefore, for a 10-kip load at the center of the span,

$$M_{Fab} = -M_{Fba} = (10)(4.52) = 45.2 \text{ ft-kips}$$

$$H_{Fab} = -H_{Fba} = 10.37 \text{ kips}$$

$$V_{Fab} = V_{Fba} = 5.0 \text{ kips}$$

The end forces are first corrected for rotation only by assuming that the auxiliary forces F, Fig. 9.16, will prevent any translation of joints a and b. The procedure is similar to that in Example 9.8.

$$H_{ab} = 10.37 + 0.062(-15.0 - 14.9) = 8.51 \text{ kips}$$

$$H_{ca} = \frac{11.3 + 22.5}{20} = 1.69 \text{ kips}$$

$$F = 8.51 + 1.69 = 10.20 \text{ kips}$$

Multiply moments in Fig. 9.15 by 10.2/3.04 and add to moments in Fig. 9.16. The final values are given in the circle. This procedure is similar to that explained in Example 9.8.

Example 9.10

ANALYSIS OF A CONTINUOUS ARCH FRAME. The continuous arch frame shown in Fig. 9.17 will be analyzed by successive approximations for a uniform load first on span ab and then on span bc. The calculations follow the same procedure as in Examples 9.8 and 9.9 except that the corrections for the horizontal displacements of points a, b, c, and d must be made separately. Consequently, five problems must be solved for any general condition of loading, that is,

(*a*) Moments due to applied load with all Δ's $= 0$ (auxiliary forces applied at a, b, c, and d to prevent translation).
(*b*) Moments due to any displacement Δ_a.
(*c*) Moments due to any displacement Δ_b.
(*d*) Moments due to any displacement Δ_c.
(*e*) Moments due to any displacement Δ_d.

However, the relationship between the end forces and the Δ's need to be determined only once, since the value of the Δ terms can then be calculated so as to remove any set of auxiliary forces obtained in item 1.

This procedure is best explained by an example. In Fig. 9.17, the span ab is subjected to a uniform vertical load of 1 kip per horizontal foot. Auxiliary horizontal forces are applied at a, b, c, and d to prevent any translation. If rotations θ_a and θ_b are first assumed equal to zero, then, from previous solutions for a fixed-end arch, we know that the fixed-end moments and horizontal reactions are

$$M_{Fab} = +126.0 \text{ ft-kips} \qquad M_{Fba} = -126.0 \text{ ft-kips}$$

$$H_{Fab} = -H_{Fba} = 36.0 \text{ kips}$$

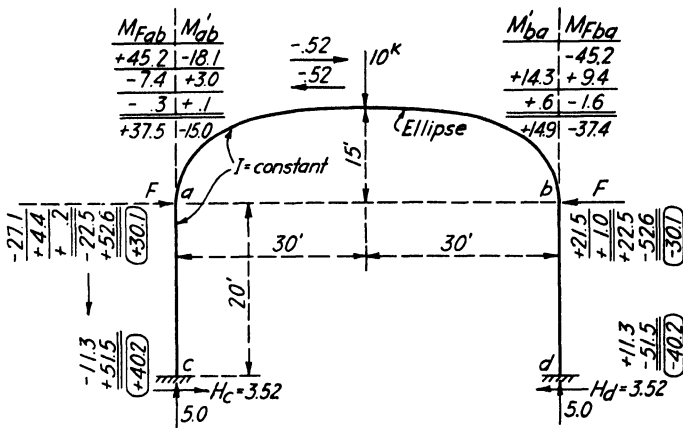



Fig. 9.16

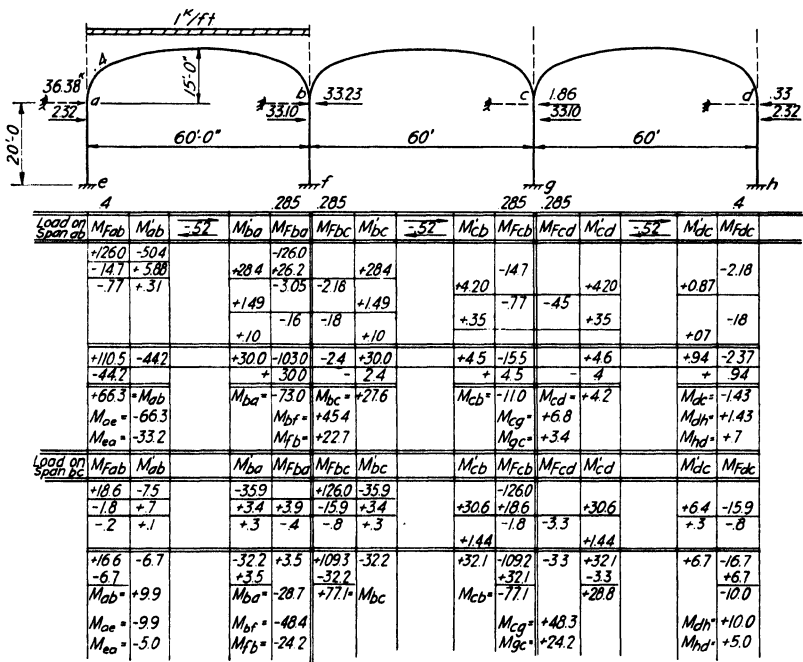



Fig. 9.17

The distribution factors for joints a and d are given in Example 9.8. For joints b and c these factors are

Member	C_b	$r = \frac{C_b}{\Sigma C_b}$	Carry-Over
ba	$0.132I$	0.285	−0.52
bc	$0.132I$	0.285	−0.52
bf	$0.20I$	0.430	+0.5
	$\Sigma C_b = 0.464I$	1.000	

The distribution of the fixed-end moments is carried out in the same manner as in Examples 9.8 and 9.9. The corrections due to rotation are kept separate, for these moments change the horizontal components, whereas the moment carried over to the other end is simply a change in the fixed-end moment. After the corrections have become sufficiently small, the final end moments can be obtained in the usual manner. The horizontal forces that act on each member must be determined before the auxiliary forces F_a, F_b, F_c, and F_d can be calculated.

The horizontal forces in the columns can, of course, be calculated directly from the end moments; the horizontal forces acting on the arch beams are equal to (see Example 9.4)

$$H_{ab} = H_{Fab} + 0.062(M_{ab}' - M_{ba}') = -H_{ba}$$

That is, the horizontal reaction on the arch is equal to the fixed-end reaction plus 0.062 times the difference of the end moments due to the rotations θ_a and θ_b. Thus, from the values recorded in Fig. 9.17,

$$H_{ab} = 36.0 + 0.062(-44.2 - 30.0) = 36.0 - 4.6 = 31.4 \rightarrow$$

$$H_e = \frac{66.3 + 33.2}{20} = 4.98 \leftarrow$$

Therefore

$$F_a = 31.4 + 4.98 = 36.38 \text{ kips} \rightarrow$$

$$H_{bc} = 0.062(30.0 - 4.5) = 1.58 \rightarrow$$

$$H_f = \frac{45.4 + 22.7}{20} = 3.41 \rightarrow$$

or

$$F_b = 31.4 + 3.41 - 1.58 = 33.23 \text{ kips} \leftarrow$$

The relationship between the end forces and any translation such as $E\,\Delta_a$ is obtained by giving $E\,\Delta_a$ any value, say 1000, while all rotations θ are kept zero. The fixed-end moments and horizontal reactions in the

Fa=1.925 a
+8.22
-15.0
-8.22
Fb=.974
b
c
Fc=.182
d
Fd=.03
-15.0
e
1.127
.328
f
.05
g
.01
h
M'ab
M'ba
M'bc
M'cb
M'cd
M'dc
-.52
He = (10.03+12.51)/20 = 1.127
Hf = (4.37+2.19)/20 = .328
Hg = 1.0/20 = .05
Hh = .01
Hab = .772+.062(3.32-2.89) = .798
Hba = .798
Hbc = .062(2.89-.44) = .152
Hcb = .152
Hcd = .062(.44-.08) = .02
Hdc = .02

(a)

Fa=.97
a
Δb
b
2.686
c
.804
d
.178
e
.38
f
1.45
g
.28
h
.056
M'ab
M'ba
M'bc
M'cb
M'cd
M'dc
He = 7.59/20 = .38
Hf = 29/20 = 1.45
Hg = 5.6/20 = .28
Hh = 1.11/20 = .056
Hab = -.772+.062(3.37-.44) = -.590
Hba = .59
Hbc = .772+.062(.44-2.47) = .646
Hcb = .646
Hcd = .062(2.47-.50) = .122
Hdc = .122

(b)

Fig. 9.18

arch and column are the same as recorded in Example 9.8. That is, for $E\,\Delta_a = 1000$ units to the right,

$$M_{Fab} = +8.22 \qquad M_{Fba} = -8.22$$
$$M_{Fae} = M_{Fea} = -15.0 \qquad H_{Fab} = 0.772 = -H_{Fba}$$

These fixed-end moments (Fig. 9.18*a*) are then distributed in the usual manner and horizontal end forces computed. The forces F_a', F_b', F_c', and F_d' that are required to make $E\,\Delta_a = 1000$ while preventing any horizontal motion of b, c, and d are now known. Both the distribution of the end moments and the auxiliary forces are recorded in Fig. 9.18*a*.

In the same manner the end moments and auxiliary forces F_a'', F_b'', F_c'', and F_d'' are obtained in terms of any displacement $E\,\Delta_b$. These values are recorded in Fig. 9.18*b*. By symmetry the values for $E\,\Delta_c$ and $E\,\Delta_d$ can be obtained by interchanging b and c, also a and d. Small variations in the auxiliary forces have been made to keep the coefficients symmetrical about the main diagonal.

To remove the auxiliary forces F when the uniform load is acting on span ab, the horizontal translations must be such that

$$\begin{aligned}
1.925\,\Delta_a - 0.974\,\Delta_b - 0.182\,\Delta_c - 0.03\,\Delta_d &= -F_a = -36.38\\
-0.974\,\Delta_a + 2.686\,\Delta_b - 0.804\,\Delta_c - 0.182\,\Delta_d &= -F_b = +33.23\\
-0.182\,\Delta_a - 0.804\,\Delta_b + 2.686\,\Delta_c - 0.974\,\Delta_d &= -F_c = +1.86\\
-0.03\,\Delta_a - 0.182\,\Delta_b - 0.974\,\Delta_c + 1.925\,\Delta_d &= -F_d = +0.33
\end{aligned}$$

The solution of these equations gives

$$E\,\Delta_a = -14.5 \qquad E\,\Delta_b = +8.1 \qquad E\,\Delta_c = +2.9 \qquad E\,\Delta_d = +2.2$$

Therefore the actual moments are equal to (moments in Fig. 9.17) + (−14.5) (moments in Fig. 9.18*a*) + (8.1) (moments in Fig. 9.18*b*) + (2.9) (moments in Fig. 9.18*b* reversed) + (2.2) (moments in Fig. 9.18*a* reversed)

The final moments for the load on span ab are given in Fig. 9.19*a*; Fig. 9.19*b* gives the values for the load on span bc.

9.13 Elastic Center Method

If the boundary forces of the fixed-end arch in Fig. 9.20 are assumed to be applied at some arbitrary point O and if the direction of the X' axis is given by some variable ϕ, the fundamental equations 9.9*a*, *b*, and *c* when applied to the V and X' axes can be simplified by choosing the proper position of O and value of ϕ. Thus, if point O and angle ϕ are selected so that

$$\sum Gh = 0 \qquad \sum Gy' = 0 \qquad \sum Ghy' = 0 \tag{9.18}$$

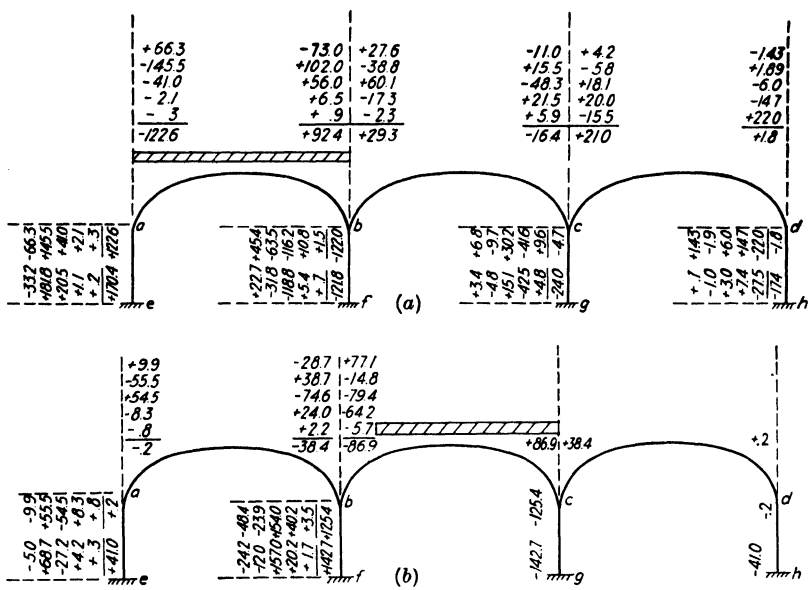



Fig. 9.19

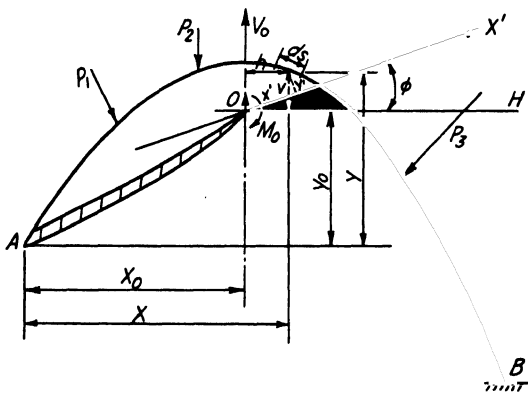



Fig. 9.20

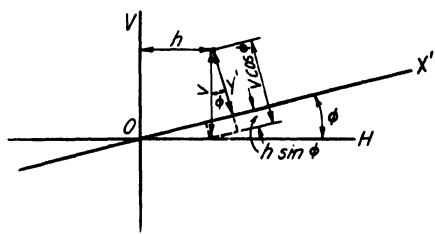



Fig. 9.21

equations 9.9*a*, *b*, and *c* reduce to

$$E\,\Delta_{ox'} = X'_o \sum Gy'^2 - \sum M'Gy' \tag{9.19a}$$

$$E\,\Delta_{ov} = V_o \sum Gh^2 + \sum M'Gh \tag{9.19b}$$

$$E\,\theta_o = M_o \sum G + \sum M'G \tag{9.19c}$$

As before, M' is positive when the applied forces on the left side of the element cause a clockwise bending moment and will therefore be negative for downward vertical loads. Since

$$h = x - x_o \qquad \text{and} \qquad v = y - y_o$$

then from equations 9.18

$$\sum Gh = \sum Gx - x_o \sum G = 0$$

or

$$x_o = \frac{\sum Gx}{\sum G} \tag{9.20a}$$

Similarly (see Fig. 9.21),

$$y' = v \cos\phi - h \sin\phi$$

Therefore

$$\sum Gy' = \cos\phi \sum Gv - \sin\phi \sum Gh$$

If $\Sigma\,Gh$ is first reduced to zero value by equations 9.20*a*, then $\Sigma\,Gy'$ will be zero if $\Sigma\,Gv = 0$. However, if

$$\sum Gv = \sum Gy - y_o \sum G = 0$$

then

$$y_o = \frac{\sum Gy}{\sum G} \tag{9.20b}$$

To make $\Sigma\,Ghy' = 0$ we will first set

$$\sum Ghy' = \sum Gh\,(v\cos\phi - h\sin\phi) = 0$$

and therefore

$$\cos\phi \sum Ghv - \sin\phi \sum Gh^2 = 0$$

or

$$\tan\phi = \frac{\sum Ghv}{\sum Gh^2} \tag{9.20c}$$

The location of point O is now known from the coordinates x_o, y_o with respect to point A and the direction of the X' axis with respect to the horizontal is given by the angle ϕ. Point O is commonly called the "elastic center," because it is located at the centroid of the elastic weights G; and axes X' and V are the principal axes for the G value.

For a symmetrical arch or frame $\Sigma\, Ghv = 0$, and therefore the X' axis is horizontal. After the new coordinate system is located, X_o', V_o, and M_o can be calculated directly from equations 9.19*a*, *b*, and *c*, or the indirect method for influence lines can be used. If an auxiliary force $F_{ox'}$ is applied at point O in the direction of the X' axis, so as to cause a displacement (see equation 9.19*a*),

$$E\,\Delta_{ox'} = F_{ox'} \sum Gy'^2$$

from equations 9.19*b* and *c*, the displacements Δ_{ov} and θ_o are zero. Since the moment at any element is equal to $F_{ox'}y'$, an influence diagram for the reaction X_o' can be obtained in the usual manner from a funicular polygram constructed by using values of $GF_{ox'}y'$ as vectors, and $E\,\Delta_{ox'}$ as the pole distance of the vector diagram. Influence diagrams for V_o and M_o can be constructed in a similar manner by using an auxiliary force F_{ov}' in the direction of V, and a moment M_o' in the direction of M_o, respectively. The pole distances for constructing these diagrams are $E\,\Delta_{ov}$ and $E\theta_o$.

Example 9.11 The frame in Example 9.9 will be analyzed by the elastic center method. The position of the horizontal axis OX' will be obtained by taking moments of the $\Delta s/I$ values, or rather the Δs values as I is constant, about a line ab. As each of the eleven elements in the arch has a length of 6.8 (Fig. 9.22),

$$\bar{y} = \frac{6.8 \sum y - (2)(20)(10)}{(11)(6.8) + (2)(20)} = \frac{(6.8)(117) - 400}{114.8} = 3.44$$

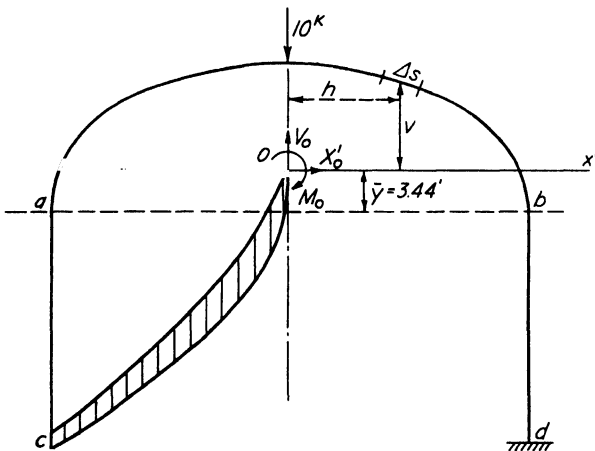



Fig. 9.22

The curved member ab will be divided into the same number of sections as in previous examples, so that

$$v = y' = y - 3.44$$

From the data in Example 9.3, the values of M' for the concentrated load of 10 kips at the center of the span and the values of y for the elements of the arch member ab (taking $G = 6.8/I =$ unity) give the following summation.

$$M'Gy' = -[(65.5)(11.16) + (131.0)(10.06) + (193.5)(8.06) + (251.0)(4.76) + (293.0)(-0.24)] = -4733$$

For the column bd, the value of $M'Gy'$ equals

$$(20/6.8)(-300.0)(-13.44) = +11{,}850$$

Total $\Sigma\, M'Gy' = -4733 + 11{,}850 = +7117$

$$\sum Gy'^2 = 11.56^2 + 2\left(11.16^2 + 10.06^2 + 8.06^2 + 4.76^2 + (-0.24)^2 + \frac{1}{6.8}\int_{3.44}^{23.44} y^2\,dy\right) = 2022$$

$$\sum M'G = -934 + \frac{(-300)(20)}{6.8} = -1814$$

$$\sum G = 11 + \frac{40}{6.8} = 16.88$$

From equation 9.19a,

$$X_o'(2022) - 7117 = 0$$

$$X_o' = H = \frac{7117}{2022} = 3.52$$

From equation 9.19b

$$V_o \sum Gh^2 - \sum M'Gh = 0$$

but, since $M' = -10h$ for half of the structure,

$$V_o \sum Gh^2 - \frac{10 \sum Gh^2}{2} = 0$$

$$V_o = 5.0 \text{ kips}$$

From equation 9.19c,

$$M_o(16.88) - 1814 = 0$$

$$M_o = 107.7 \text{ ft-kips}$$

The moment at support c is

$$M_c = 107.7 - (5)(30) + (3.52)(23.44) = 40.2 \text{ ft-kips}$$

9.14 *Analysis of Stiff Rings and Closed Frames*

The analysis of stiff rings and closed frames as illustrated in Figs. 9.23*a* and *b* can be made directly from equations 9.9*a*, *b*, and *c* as for an arch. In fact, such structures can be regarded as arches whose abutments coincide. When the equation of the axis of the ring is known and the moment of inertia is constant, the summations can be replaced by integrals. However, for a general solution adaptable to all conditions, the use of finite increments of length is recommended. Such structures are frequently symmetrical about the vertical axis, which greatly reduces the numerical work. The signs of the various terms are readily obtained by inspection, and they should be checked before the equations are solved. A numerical example will illustrate the application of the equations to the analysis of stiff rings.

Example 9.12 The circular ring of constant I in Fig. 9.24 is subjected to concentrated loads which are symmetrical with respect to the vertical axis. The ring is cut at the top, so that each side can be regarded as a fixed support with equal but opposite forces. Each half of the ring is divided into twelve equal divisions whose G or $\Delta s/I$ value can be taken as unity. Bending moments that produce tension on the inside of the ring are regarded as positive. Therefore the unknowns M_0, H_0, and V_0 should be given positive values in the equations and M' negative. Table 9.4 (p. 328) gives the numerical value of the summations for G and radius r both equal to unity. If the G values are not constant, they must, of course, be included in each term of the summation.

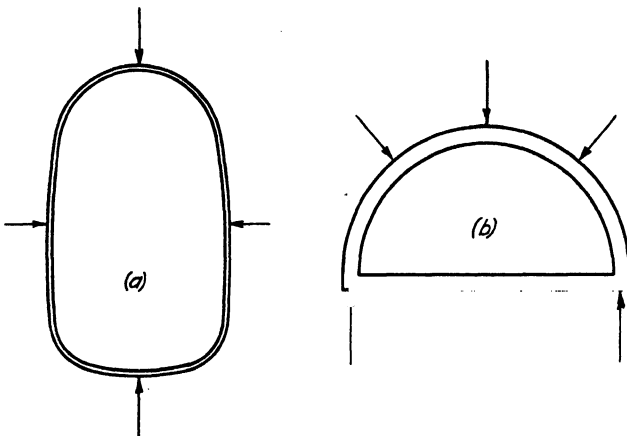



Fig. 9.23

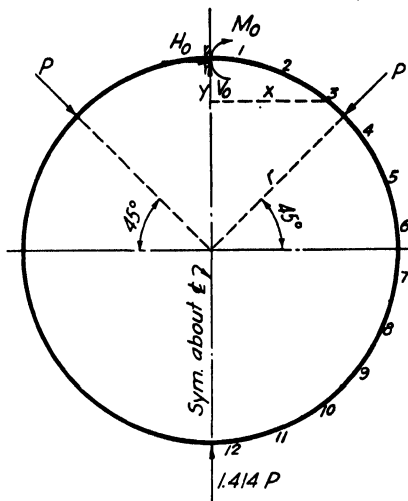



Fig. 9.24

The values of the summation terms for the entire ring are

$$\sum G = 24$$
$$\sum Gx = \sum Gxy = 0$$
$$\sum Gy = 24.00$$
$$\sum Gx^2 = 12.0$$
$$\sum Gy^2 = 36.0$$
$$\sum M'G = -13.0788$$
$$\sum M'Gx = 0$$
$$\sum M'Gy = -19.46$$

Rewriting equations 9.9*a*, *b*, and *c* for positive y downward and taking all the unknowns M_0, H_0, and V_0 as positive quantities give

$$24.0M_0r + V_0(0)r^2 + 36.0H_0r^2 - 19.46Pr^2 = 0$$
$$0M_0r + 12.0V_0r^2 + (0)(H_0r^2) - 0 = 0$$
$$24.0M_0 + (0)V_0r + 24.0H_0r - 13.079Pr = 0$$

from which

$$V_0 = 0 \qquad H_0 = 0.532P \qquad M_0 = 0.012Pr$$

The moment at the base is therefore

$$M = 0.012Pr + (2r)(0.532P) - 0.707Pr = 0.369Pr$$

At the concentrated loads P,

$$M = 0.012Pr + (0.293r)(0.532P) = 0.168Pr$$

Table 9.4

(For right half of ring; left half is identical except that x is negative)

Section	x	y	x^2	y^2	M'	$M'x$	$M'y$
1	0.1305	0.0086	0.016	....	0	0	0
2	0.3827	0.0761	0.147	0.005	0	0	0
3	0.6088	0.2066	0.370	0.043	0	0	0
4	0.7934	0.3912	0.630	0.153	−0.1305	−0.104	−0.051
5	0.9239	0.6173	0.854	0.382	−0.3827	−0.353	−0.236
6	0.9914	0.8695	0.983	0.756	−0.6088	−0.604	−0.530
7	0.9914	1.1305	0.983	1.278	−0.7934	−0.787	−0.897
8	0.9239	1.3827	0.854	1.912	−0.9239	−0.854	−1.280
9	0.7934	1.6088	0.630	2.588	−0.9914	−0.787	−1.596
10	0.6088	1.7934	0.370	3.216	−0.9914	−0.604	−1.780
11	0.3827	1.9239	0.147	3.701	−0.9239	−0.353	−1.780
12	0.1305	1.9914	0.016	3.966	−0.7934	−0.104	−1.580
Total		12.00	6.000	18.000	−6.5394	−4.550	−9.730

9.15 Fuselage Frames

In the types of structural framing frequently used in aircraft construction, the applied loads from wings, tail surfaces, landing gear, water pressure, and various other sources are commonly passed into the hull or fuselage through main frames that are fairly stiff. These applied forces are resisted by shearing stresses between the outer edge of the frame and the skin of the fuselage (Fig. 9.25). The distribution of these shearing forces is a statically indeterminate problem in itself and is frequently

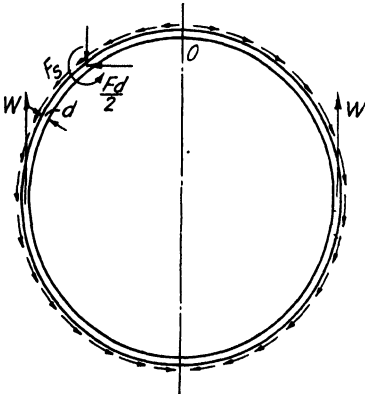



Fig. 9.25

approximated by one of the following assumptions.

1. The shearing stresses are distributed as for any beam section, that is, by the formula $q = VQ/I$, in which Q and I depend on the arrangement of effective longitudinal flange area of the fuselage.
2. The shearing stresses are proportional to their distance from the applied load.

In general, it would seem that the second assumption will be best if the applied loads are concentrated at one or two points and that the first assumption should be used if the applied load is distributed. Because the interaction between the frames and the skin of the fuselage is exceedingly complex, an exact solution is not feasible. Therefore any analysis that is based on an assumed two-dimensional force system must be an approximate solution, but should give conservative values for the stresses.

The analysis of any fuselage frame, such as in Fig. 9.25, is readily made by equations 9.9 as for a ring. The Δs values, as well as the coordinates of each element, can be obtained graphically if the frame is drawn to a large scale. For convenience the shearing forces which are tangent to the outer edge can be replaced at the axis of the frame by horizontal and vertical components and a couple $Fd/2$. If the origin is taken on the axis at point O at the top of the frame, the solution can be arranged in table form as in Example 9.12.

If the loads are applied through members that are monolithic with the frame (Fig. 9.26*a*), the following procedure is recommended.

1. Calculate the fixed-end moments and the horizontal reactions for the separate portions OAO', OBO', and OO' as shown in Fig. 9.26*b*.

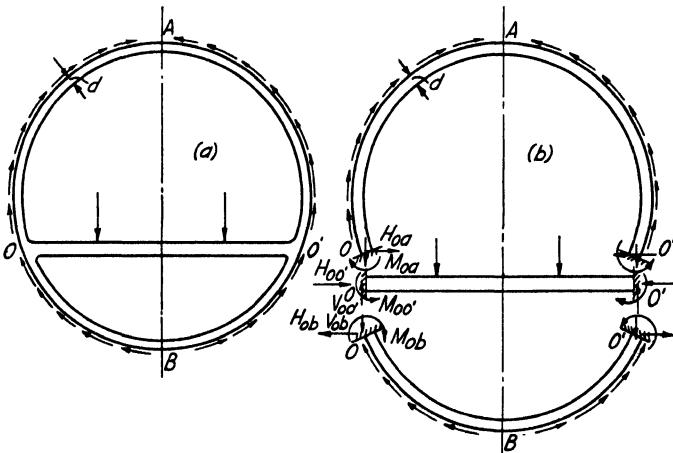



Fig. 9.26

2. Determine the distribution and carry-over factors for each member. (See Articles 9.11 and 9.12.)
3. Distribute the unbalanced moments and record the necessary corrections to the fixed-end moments and the horizontal reactions.

The analysis will be illustrated by a solution of the frame in Fig. 9.27.

Example 9.13 The calculation of the bending moments at points O and O' (Fig. 9.27) by the method previously outlined requires the magnitude and distribution of the shearing forces along the outer edge of the frame. The second assumption will be used, that is, that the shearing force per linear foot is proportional to its distance from point O and is zero at the top and bottom of the frame. Accordingly, if q' is the unit shearing force at point O, then q, the unit shearing force at any point, is

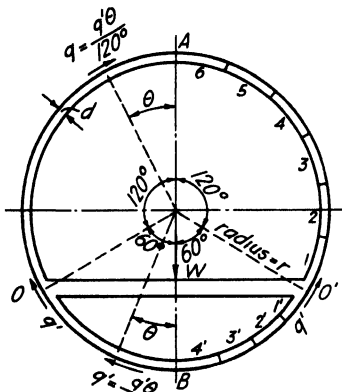



Fig. 9.27

$$q = \frac{q'\theta}{120} \qquad \text{for the portion } OA$$

$$q = \frac{q'\theta}{60} \qquad \text{for the portion } OB$$

where θ is measured from A for OA and from B for OB.

The tangential shearing force on any arc Δs will therefore be

$$F_s = q\,\Delta s = \frac{q'\theta}{120} \quad \Delta s \text{ for portion } OA$$

and

$$F_s = q\,\Delta s' = \frac{q'\theta}{60} \quad \Delta s' \text{ for portion } OB$$

The horizontal and vertical components of F_s are

$$F_{sx} = F_s \cos\theta$$

$$F_{sy} = F_s \sin\theta$$

The portion OA will be divided into six equal divisions numbered 1 to 6, and OB into four divisions numbered 1′ to 4′. Table 9.5 gives the calculations for F_{sx}, F_{sy}, and q'.

From the equilibrium condition

$$\sum_{1}^{6} F_{sy} + \sum_{1'}^{4'} F_{sy} = \frac{W}{2}$$

Table 9.5

Point	θ	Shearing Force $F_s = \frac{\theta}{120} q' \Delta s$	$F_{sx} = F_s \cos\theta$	$F_{sy} = F_s \sin\theta$
6	10°	$\frac{1}{12} q' \Delta s$	$0.082\, q' \Delta s$	$0.0145\, q' \Delta s$
5	30°	$\frac{3}{12}$	0.216	0.1250
4	50°	$\frac{5}{12}$	0.268	0.3192
3	70°	$\frac{7}{12}$	0.199	0.547
2	90°	$\frac{9}{12}$	0	0.750
1	110°	$\frac{11}{12}$	-0.305	0.861
			$\sum_1^6 F_{sx} = 0.460\, q' \Delta s$	$\sum_1^6 F_{sy} = 2.617\, q' \Delta s$
4′	7.5°	$\frac{1}{8} q' \Delta s'$	$-0.124\, q' \Delta s'$	$0.0163\, q' \Delta s'$
3′	22.5°	$\frac{3}{8}$	-0.346	0.1438
2′	37.5°	$\frac{5}{8}$	-0.495	0.380
1′	52.5°	$\frac{7}{8}$	-0.533	0.694
			$\sum_{1'}^{4'} F_{sx} = -1.498\, q' \Delta s'$	$\sum_{1'}^{4'} F_{sy} = 1.234\, q' \Delta s'$

and

$$\Delta s = \frac{\pi r}{9} \qquad \Delta s' = \frac{\pi r}{12}$$

we obtain

$$2.617q'\left(\frac{\pi r}{9}\right) + 1.234q'\left(\frac{\pi r}{12}\right) = \frac{W}{2}$$

from which

$$q' = 0.404\,\frac{W}{r} \qquad q'\,\Delta s = 0.141\,W \qquad q'\,\Delta s' = 0.106W$$

Therefore all shearing forces and components can be obtained by substituting the values of $q'\ \Delta s$ and $q'\ \Delta s'$ in Table 9.5. If the depth of the frame is d in., the couple acting at each point will be $F_s d/2$. The vertical component at O acting on the portion OAO' is $(2.617)(0.141W)$, or $0.369W$, and that on the portion OBO' is $(1.234)(0.106W)$, or $0.131W$.

Since the vertical reactions are now known, they can be considered part of the applied load when computing M' in equations 9.9*a* and *c*, that is,

$$E\,\Delta_{ox} = -M_{oa}\sum Gy + H_{oa}\sum Gy^2 - \sum M'Gy = 0$$

$$E\theta_{oa} = M_{oa}\sum G - H_{oa}\sum Gy + \sum M'G = 0$$

where M' is the moment at any point with M_{oa} and H_{oa} removed. Only half the structure need be used, for the summations in these equations for the entire structure are twice that for one-half.

Table 9.6 Data for Portion OAO′

(Origin at O)

Point	x	y	y^2	M'	$M'y$
1	$-0.0737r$	$0.158r$	$0.0250r^2$	$0.0272Wr + 0.065Wd$	$0.0043Wr^2 + 0.010Wdr$
2	$-0.134r$	$0.500r$	$0.2500r^2$	$0.0573Wr + 0.118Wd$	$0.0286Wr^2 + 0.059Wdr$
3	$-0.0737r$	$0.842r$	$0.709r^2$	$0.0639Wr + 0.158Wd$	$0.0538Wr^2 + 0.133Wdr$
4	$+0.1000r$	$1.1428r$	$1.306r^2$	$0.0575Wr + 0.188Wd$	$0.0657Wr^2 + 0.215Wdr$
5	$+0.366r$	$1.366r$	$1.866r^2$	$0.0475Wr + 0.206Wd$	$0.0649Wr^2 + 0.282Wdr$
6	$+0.6924r$	$1.4848r$	$2.205r^2$	$0.0406Wr + 0.211Wd$	$0.0603Wr^2 + 0.313Wdr$
Σ	$0.877r$	$5.4936r$	$6.361r^2$	$0.2940Wr + 0.946Wd$	$0.2776Wr^2 + 1.012Wdr$

After substituting the values from Table 9.6 into the preceding equations for $E\,\Delta_{ox}$ and $E\theta_{oa}$, we obtain (for $G = 1$)

$$-M_{oa}(5.494r) + 6.361r^2H_{oa} = +0.2776Wr^2 + 1.012W\,dr$$

$$M_{oa}(6.00) - 5.494rH_{oa} = -0.2940Wr - 0.946W\,dr$$

from which

$$H_{oa} = +0.0063W + 0.1095\frac{Wd}{r} \qquad + \rightarrow$$

$$M_{oa} = -0.0432Wr - 0.057Wd \qquad +\circlearrowright$$

In the same manner the fixed-end reactions at O and O' for the portion OBO' are determined from the summations in Table 9.7.

Table 9.7 Data for Portion OBO′

(Origin at O)

Point	x	y	y^2	M'	$M'y$
1′	$0.0726r$	$0.1088r$	$0.0118r^2$	$-0.0095Wr + 0.0066Wd$	$-0.00103Wr^2 + 0.00072Wdr$
2′	$0.2572r$	$0.2934r$	$0.0860r^2$	$-0.0305Wr + 0.0265Wd$	$-0.00894Wr^2 + 0.00776Wdr$
3′	$0.4833r$	$0.4329r$	$0.1800r^2$	$-0.0488Wr + 0.0595Wd$	$-0.02074Wr^2 + 0.02525Wdr$
4′	$0.7355r$	$0.4914r$	$0.2420r^2$	$-0.0591Wr + 0.106Wd$	$-0.02910Wr^2 + 0.05215Wdr$
Σ		$1.3175r$	$0.5198r^2$	$-0.1479Wr + 0.1986Wd$	$-0.05981Wr^2 + 0.08588Wdr$

$$1.318M_{ob}r + 0.520H_{ob}r^2 = 0.0598Wr^2 - 0.0859Wdr$$

$$4.00M_{ob} + 1.318H_{ob}r = 0.1479Wr - 0.1986Wd$$

from which

$$H_{ob} = 0.1293W - 0.237\frac{Wd}{r} \qquad + \rightarrow$$

$$M_{ob} = -0.0056Wr + 0.0285Wd \qquad +\circlearrowright$$

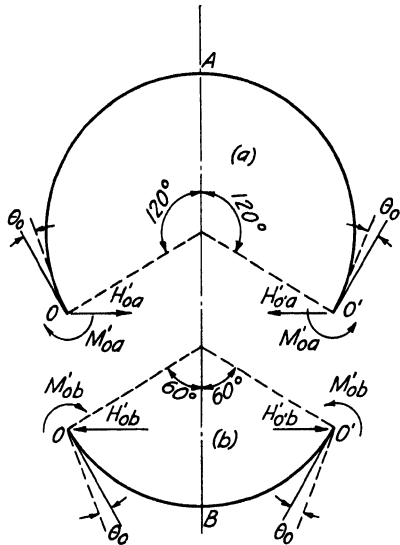



Fig. 9.28

For the beam OO' the fixed-end moments for a concentrated load W at the center is

$$M_{Foo'} = -\frac{WL}{8} = -\frac{W2r \sin 60^\circ}{8} = -0.216Wr$$

Since the fixed-end forces at the joints O and O' have now been calculated for all three members, only the task of distributing the unbalanced moment to each member remains. This problem can be solved in the usual way by rotating the joints O and O' separately, but the easiest procedure is to take advantage of symmetry and rotate both joints at the same time and thereby eliminate the carry-over procedure. The correction is then made in one operation.

If two equal and opposite moments M_{oa}' are applied to a symmetrical curved member OAO' (Fig. 9.28*a*) and the ends are restrained against horizontal displacement, from equations 9.9*a* and *c*,

$$E\,\Delta_{oa} = -M_{oa}' \sum Gy + H_{oa}' \sum Gy^2 = 0$$

from which

$$H_{oa}' = \frac{\sum Gy}{\sum Gy^2} M_{oa}' \qquad (9.21a)$$

and

$$2E\theta_o = M_{oa}' \sum G - H_{oa}' \sum Gy$$

or

$$M_{oa}' = \frac{2E\theta_o}{\sum G - \dfrac{(\sum Gy)^2}{\sum Gy^2}} \tag{9.21b}$$

Since G represents $\Delta s/I$ for each element, these equations can be used for frames with variable I. For the portion OAO', G equals a constant $\pi r/9I$ for all elements and therefore (values from Table 9.5)

$$M_{oa}' = \frac{2E\theta_o}{2\,\dfrac{\pi r}{9I}\left[6.0 - \dfrac{(5.494)^2}{6.361}\right]} = 2.26\,\frac{EI}{r}\,\theta_o$$

and

$$H_{oa}' = \frac{5.494}{6.361}\,\frac{M_{oa}'}{r} = 0.863\,\frac{M_{oa}'}{r}$$

acting to the right for clockwise moment.

For the portion OBO',

$$M_{ob}' = \frac{2E\theta_o}{2\,\dfrac{\pi r}{12I}\left[4.0 - \dfrac{(1.318)^2}{0.520}\right]} = 5.78\,\frac{EI}{r}\,\theta_o$$

and

$$H_{ob}' = \frac{1.318}{0.520}\,\frac{M_{ob}'}{r} = 2.53\,\frac{M_{ob}'}{r}$$

acting to the left for clockwise moment.

For the beam OO', the relations are

$$M_{oo'} = \frac{EI}{L}\,(4\theta_o + 2\theta_o') = \frac{2EI}{L}\,\theta_o$$

or, since

$$L = (2)(0.866r)$$

$$M_{oo'} = 1.15\,\frac{EI}{r}\,\theta_o$$

If the rotation θ_o is assumed to be the same for both frame and beam, the distribution factors are

For OAO'

$$\frac{2.26I_f}{2.26I_f + 5.78I_f + 1.15I_b}$$

where I_f = moment of inertia of frame

I_b = moment of inertia of beam

For OBO'

$$\frac{5.78I_f}{2.26I_f + 5.78I_f + 1.15I_b}$$

For OO'

$$\frac{1.15I_b}{2.26I_f + 5.78I_f + 1.15I_b}$$

If we let $I_b = 10I_f$, the distribution factors are

For OAO'	0.116
For OBO'	0.296
For OO'	0.588
	1.000

Let $r = 60$ in., $d = 5$ in.
Fixed-end moments are

$$M_{Foa} = (-0.0432W)(60) - (0.057W)(5) = -2.592W - 0.285W$$
$$= -2.877W \text{ in.-lb}$$
$$M_{Fob} = -0.0056W(60) + (0.0285W)(5) = -0.194W \text{ in.-lb}$$
$$M_{Foo}' = -0.216W(60) = -12.96W$$
$$\sum M_{Fo} = -2.87W - 0.19W - 12.96W = -16.02W$$

Therefore the actual end moments are

$$M_{oa} = -2.87W + (0.116)\,(16.02W) = -1.01W$$
$$M_{ob} = -0.19W + (0.296)\,(16\;02W) = +4.55W$$
$$M_{oo}' = 12.96W + (0.588)\,(16.02W) = -3.54W$$

The final values of H are

$$H_{oa} = +0.0063W + (0.1095W)\frac{5}{60} + \frac{(0.863)(1.86\quad)}{60} = 0.0322W \rightarrow$$

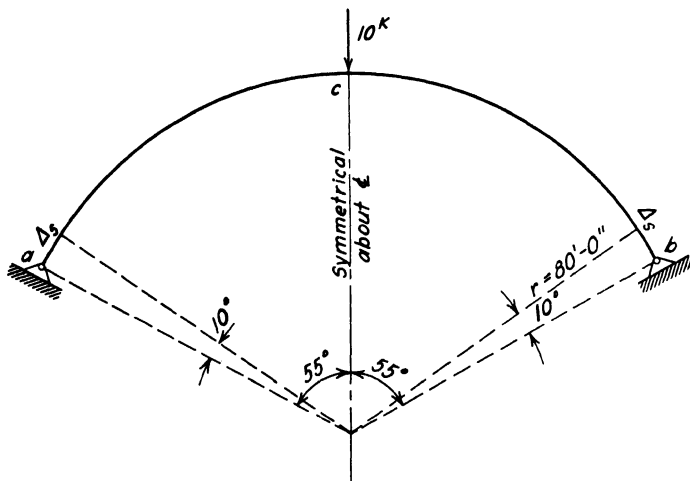
$$H_{ob} = +0.1293W - (0.237W)\frac{5}{60} - \frac{(2.53)(4.74W)}{60} = -0.0905W \leftarrow$$

$$H_{oo'} = +0.0905 - 0.0322W = 0.0583W \rightarrow$$

Problems

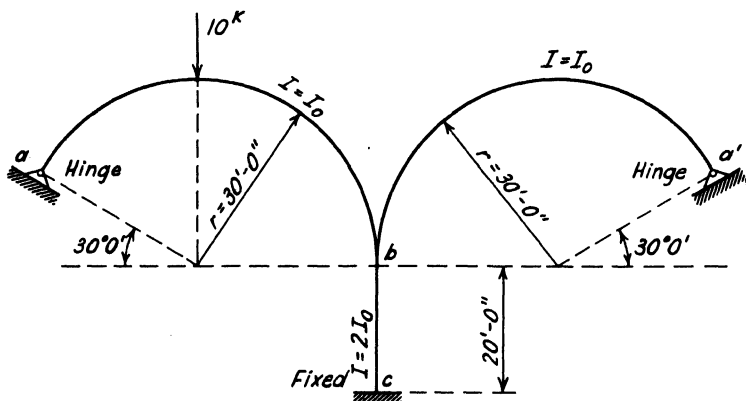
9.1 (*a*) Calculate the horizontal reaction at the hinges *a* and *b* and the bending moment at the crown *c* for the two-hinged circular arch of constant cross section. Divide the arch axis into eleven equal elements Δs.

(*b*) Compare the values in part *a* with the corresponding values when integration instead of summation is used.

(*c*) Determine the vertical displacement of point *c* in terms of *EI*. Use the summation of angle changes times horizontal distance and then check by Castigliano's theorem using integration.

Problem 9.1

9.2 Construct an influence diagram for the horizontal reaction of the two-hinged arch in Problem 9.1 for a unit vertical load. Use the indirect method.

9.3 Construct influence diagrams for the horizontal reaction at point *a* and the end couple at point *b* for the arch axis of Example 9.1 if end *a* is hinged and end *b* is fixed. Use the Δs, x, and y values tabulated in Table 9.1. Consider a unit vertical load and assume *EI* constant.

9.4 Calculate the reactions for the members *ab*, *ba'*, and *bc* for the two-span continuous arch shown. Assume that the *I* of the column *bc* is twice

Problem 9.4

the I of the arch members and that E is constant. See Problem 9.3 and Example 9.1 for dimensions of the arches. First write the deformation equations in terms of $E\theta_b$ and $E\Delta_b$ and solve the problem by either moment distribution or slope deflection.

9.5 Construct influence diagrams for H_a, V_a, and M_a for the arch in Example 9.1 if the arch is fixed at both supports.

9.6 The unsymmetrical reinforced concrete arch rib shown has the following properties:

$$\text{constant width} = 36 \text{ in.,} \qquad y_1 = \frac{x^2}{60}, \qquad y_2 = \frac{x^2}{80}$$

$$t = 19 + 0.025x^{1.8}$$

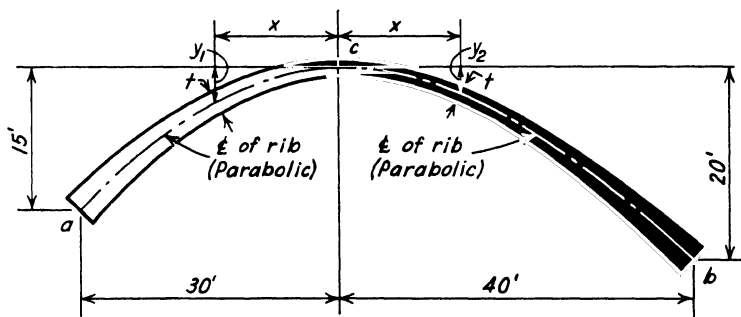
where t = thickness in inches

x = horizontal distance in feet from point c

y = vertical distance in feet from point c

use $E = 3.5 \times 10^6$ psi

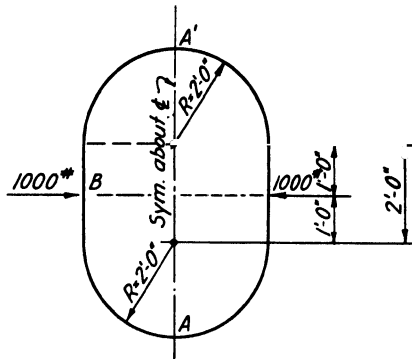
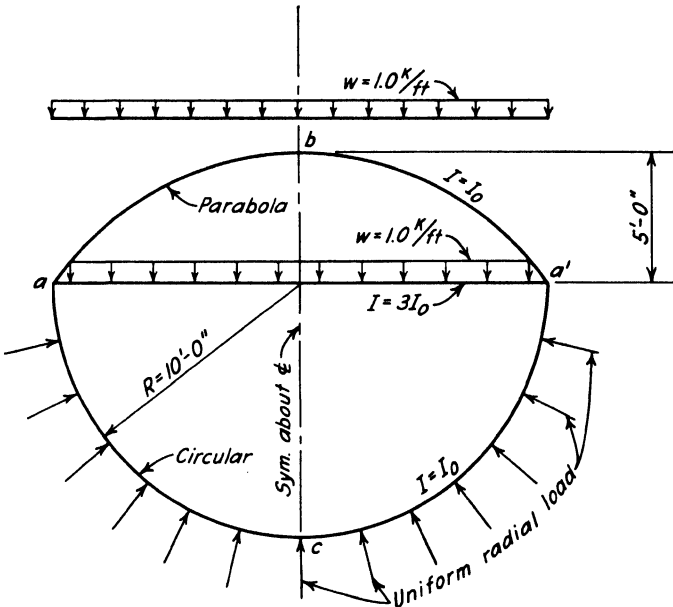
(*a*) Construct influence diagrams for H_a, V_a, and M_a, for a unit vertical load if the arch is fixed at both ends. Use the indirect method and consider the influence of the normal forces on the horizontal motion of point a.

(*b*) Repeat part a for a unit horizontal load.

Problem 9.6

9.7 Using the arch rib described in Problem 9.6, calculate the reactions at the fixed supports for a vertical load of 15 kips at point c by the elastic center method. Check your answers with the results obtained in Problem 9.6.

9.8 Determine the reactions and the bending moment at point c for the arch in Problem 9.6 for a temperature drop of 60°F for the following conditions: (*a*) both supports fixed and (*b*) both supports hinged.
Assume a coefficient of linear expansion of 6×10^{-6} per °F.

9.9 What are the internal forces acting at points A and B in the frame shown? Assume EI constant and consider flexural action only.

9.10 Calculate the reactions for the members *aba'*, *aca'* and *aa'* if the arch members are continuous and if the horizontal member *aa'* is (*a*) hinged at the ends and (*b*) monolithic with the arch members.
Use the relative I values given on the diagram and assume E constant.

Problem 9.9

Problem 9.10

References

1 McCullough and Thayer, *Elastic Arch Bridges*, John Wiley and Sons.

2 Parcel and Moorman, *Analysis of Statically Indeterminate Structures*, John Wiley and Sons.

3 Hool, *Reinforced-Concrete Construction*, Vol. III, McGraw-Hill Book Co.

4 Cross and Morgan, *Continuous Frames of Reinforced Concrete*, John Wiley and Sons.

5 Alexander Hrennikoff, "Analysis of Multiple Arches," *Trans. Am. Soc. C.E.*, Vol. 10 (1936).

6 J. A. Wise, "Segmental Division for Arch Analysis," *Civil Engg.*, January, 1938.
7 J. A. Wise, "Analysis of Circular Rings for Monocoque Fuselages," *J. Inst. Aero. Sci.*, September, 1939.
8 B. F. Ruffner, Jr., "Monocoque Fuselage Circular Ring Analysis," *J. Inst. Aero. Sci.*, January, 1939.
9 J. M. Garrelts, "Design of St. Georges Tied Arch Span," *Trans. Am. Soc. C.E.*, Vol. 108 (1943).
10 J. C. Rathbun, "An Analysis of Multiple-Skew Arches on Elastic Piers," *Trans. Am. Soc. C.E.*, Vol. 98 (1933).
11 Charles S. Whitney, "Design of Symmetrical Concrete Arches," *Trans. Am. Soc. C.E.*, Vol. 88 (1925).
12 Charles S. Whitney, "Analysis of Unsymmetrical Concrete Arches," *Trans. Am. Soc. C.E.*, Vol. 99 (1934).
13 Michalos, *Theory of Structural Analysis and Design*, Ronald Press.

10

Flexible Members

10.1 Introduction

In the preceding chapters various structures have been analyzed for the condition that the deflections are sufficiently small, as compared to the overall dimensions, so that they can be neglected when calculating stresses and displacements. As the length-depth ratios of the members increase, however, proportions are reached where such an assumption is not permissible. The error involved in neglecting the displacement of the axes of the members in flexible structures may lead to either dangerous conclusions or an uneconomical design. It therefore seems essential that the engineer be familiar with such structures, at least to the extent of recognizing the problems when they occur. At the same time, it will often be expedient to avoid the mathematically complex exact solutions by substituting a solution by successive approximations or even an approximate one. The fundamental equations must, of course, always remain the basis for judging the validity of any short cut in the numerical calculations. A number of statically indeterminate problems in which the deflection is important will now be considered to illustrate both the fundamental equations and the use of successive approximations.

Nonlinear relations between forces and displacements in flexible members have been previously described in Article 2.1 and in Fig. 2.2. The principle of superposition is, in general, not applicable to such structures. Whenever the deflection of the structure is included in the coordinate

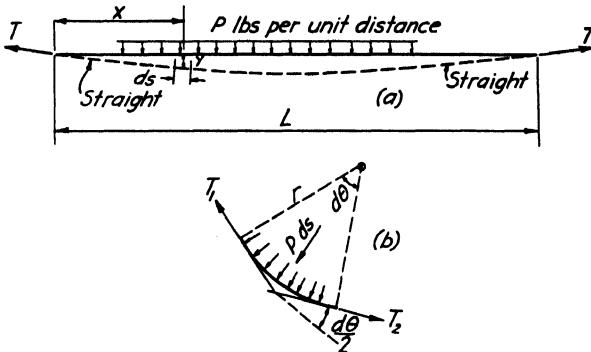



Fig. 10.1

system the deflection theory as contrasted to the usual elastic theory is obtained.

10.2 Wires Subjected to Radial Pressure

When a straight member that has no flexural resistance is subjected to a radial pressure (Fig. 10.1*a*), the tension T must be constant throughout. The tension is constant because the equilibrium of any element ds (Fig. 10.1*b*) requires that

$$T_1 \cos \frac{d\theta}{2} = T_2 \cos \frac{d\theta}{2} \qquad T_1 = T_2 = T$$

and

$$\frac{p\, ds}{2T} = \sin \frac{d\theta}{2} = \frac{d\theta}{2} \qquad \text{(for small angles)} \tag{10.1a}$$

Since

$$ds = r\, d\theta$$

$$p = \frac{T}{r} = T \frac{d^2y}{dx^2} \tag{10.1b}$$

for

$$\frac{1}{r} = \frac{d^2y}{dx^2} \qquad \text{(for small curvature)}$$

Although this equation of equilibrium gives the relation between T, y, p, and L, the numerical value of T cannot be calculated directly. To determine T it is necessary to combine this equilibrium equation with the strain condition that the change in length of the wire produced by the constant tension T be consistent with the displacements necessary to provide equilibrium. The change in length of the wire ΔL resulting from the tension T is

$$\Delta L = \frac{TL}{AE} \tag{10.2a}$$

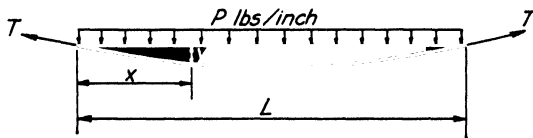



Fig. 10.2

whereas the change in length due to relatively small deflections is obtained in the following manner:

$$ds = \sqrt{dx^2 + dy^2} = dx[1 + (dy/dx)^2]^{1/2}$$

Expanding by the binomial theorem gives

$$ds = dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 - \frac{1}{8}\left(\frac{dy}{dx}\right)^4 + \dots\right]$$

If only the first two terms of the series are used, the increase in length equals

$$\Delta L = \int_0^L ds - \int_0^L dx = \frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx \tag{10.2b}$$

Equating this value to that obtained by equation 10.2*a* gives

$$\frac{TL}{AE} = \frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx \tag{10.3}$$

From equation 10.1*b*

$$\frac{dy}{dx} = \frac{1}{T}\left(\int p\, dx + C_1\right) \tag{10.4}$$

so that a numerical solution for T can be made for any distribution of p.

Example 10.1 The value of T for a constant radial pressure p over the entire span (Fig. 10.2) will be calculated.

$$T\frac{d^2y}{dx^2} = p$$

$$T\frac{dy}{dx} = px + C_1$$

$$Ty = \frac{px^2}{2} + C_1x + C_2$$

When

$$x = 0 \qquad y = 0 \qquad C_2 = 0$$

$$x = L \qquad y = 0 \qquad C_1 = -\frac{pL}{2}$$

Therefore

$$\frac{dy}{dx} = \frac{p}{T}\left(x - \frac{L}{2}\right)$$

Substituting this expression in equation 10.3 gives

$$\frac{TL}{AE} = \frac{1}{2}\int_0^L \frac{p^2}{T^2}\left(x - \frac{L}{2}\right)^2 dx$$

from which

$$T^3 = \frac{p^2L^2AE}{24}$$

If $p = 4$ lb per inch, $L = 20$ ft, $A = 0.05$ in.^{2}, and $E = 27 \times 10^6$ lb per square inch,

$$T^3 = \frac{(4)(4)(400)(144)(0.05)(27)(10^6)}{24} = (52)(10^9)$$

$$T = 1000\sqrt[3]{52} = 3730 \text{ lb}$$

or a unit stress of $3730/0.05 = 74{,}600$ lb per square inch.

If the wire has an initial tension T_0 before the pressure is applied, the change in length ΔL is due to the difference between the initial and final tension, or

$$\Delta L = \frac{(T - T_0)L}{AE} = \frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx \qquad (10.5a)$$

For a constant pressure p and initial tension T_0, we obtain

$$\frac{(T - T_0)L}{AE} = \frac{1}{2}\int_0^L \frac{p^2}{T^2}\left(x - \frac{L}{2}\right)^2 dx$$

from which

$$T^2(T - T_0) = \frac{p^2L^2AE}{24}$$

or

$$T = \frac{pL}{2}\sqrt{AE/6(T - T_0)} \qquad (10.5b)$$

This equation can be quickly solved by trial.

Let T_0 equal 800 lb, with the other values remaining as before. Then

$$\frac{pL}{2} = \frac{(4)(20)(12)}{2} = 480 \qquad AE = 135 \times 10^4$$

or

$$T = 48{,}000\sqrt{135/6(T - 800)}$$

By trial, $T = 4010$ lb. Any change in span length affected by temperature or movement of the supports can be added algebraically to ΔL.

10.3 *Wires and Cables Subjected to Vertical Loads*

When the applied loads are vertical instead of radial, the equilibrium conditions for any element ds of the cable (Fig. 10.3) require that the horizontal component H, rather than the axial tension T, remains constant between points of support. Therefore the summation of all vertical components acting on any element gives the equation

$$H\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\,dx\right) - H\frac{dy}{dx} - q\,ds = 0$$

or

$$H\frac{d^2y}{dx^2} = q\frac{ds}{dx} \tag{10.6}$$

where q represents the intensity of the vertical load distributed along the axis.

If the vertical load q is distributed along the horizontal projection, equation 10.6 takes the form

$$H\frac{d^2y}{dx^2} = q \tag{10.7}$$

In this discussion the positive directions for x and y are as indicated in Fig. 10.3. If the positive direction of y is taken downward, the right side of equation 10.6 and 10.7 will be negative. For the present it will be assumed that the weight is distributed along the axis and equation 10.6 will be used.

In his paper on the "Static and Dynamic Analysis of Guy Cables," Transactions Am. Soc. of Civil Engrs., Vol. 127, Part II, 1962, Professor Donald Dean gave the following solutions for determining the tension T and the shape of the deflected cable for static conditions.

$$\frac{d^2y}{dx^2} = \frac{q}{H}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} \tag{10.8a}$$

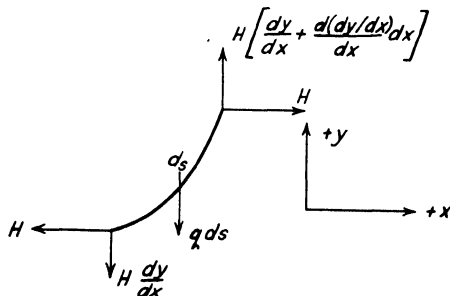



Fig. 10.3

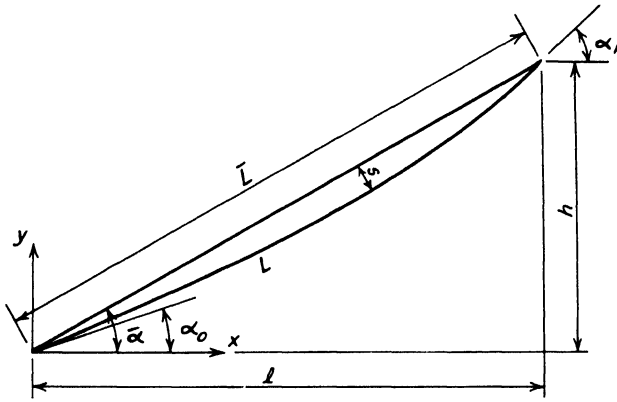



Fig. 10.4

Integrating once gives

$$\frac{dy}{dx} = \sinh\left(\frac{q}{H}x + a_1\right) \qquad (10.8b)$$

and by integrating again we obtain

$$y = \frac{H}{q}\cosh\left(\frac{q}{H}x + a_1\right) + a_2 \qquad (10.8c)$$

where a_1 and a_2 are constants. This solution can be checked by differentiating equation 10.8*c* to give equation 10.8*b* and then equation 10.8*b* to give

$$\frac{d^2y}{dx^2} = \frac{q}{H}\left[\cosh\left(\frac{q}{H}x + a_1\right)\right]$$

but

$$\cosh\left(\frac{q}{H}x + a_1\right) = \left[1 + \sinh^2\left(\frac{q}{H}x + a_1\right)\right]^{1/2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

Therefore

$$\frac{d^2y}{dx^2} = \frac{q}{H}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} = \frac{q}{H}\frac{ds}{dx}$$

For the cable shown in Fig. 10.4, the constants of integration must satisfy the boundary conditions

$$y = 0 \qquad \text{when} \qquad x = 0$$

and

$$y = h \qquad \text{when} \qquad x = l,$$

from which

$$a_2 = -\frac{H}{q}\cosh a_1 \qquad (10.9a)$$

and

$$a_1 = \sinh^{-1}\left(\frac{qh}{2H \sinh r}\right) - r \tag{10.9b}$$

in which

$$r = \frac{ql}{2H} \tag{10.9c}$$

Furthermore,

$$T = H\frac{ds}{dx} = H\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} = H \cosh\left(\frac{q}{H}x + a_1\right) \tag{10.10}$$

and the length L along the curve is

$$L = \int_0^l \frac{ds}{dx}\,dx = \int_0^l \cosh\left(\frac{q}{H}x + a_1\right) dx$$

which gives

$$L = \left(h^2 + \frac{4H^2}{q^2}\sinh^2 r\right)^{1/2} \tag{10.11a}$$

or

$$\sqrt{L^2 - h^2} = \frac{2H}{q}\sinh r \tag{10.11b}$$

The end slopes α_0 and α_1 can be determined directly from equation 10.8*b* in the form

$$\tan \alpha_0 = \left.\frac{dy}{dx}\right|_{x=0} = \sinh a_1 \tag{10.12a}$$

which can also be expressed in the form

$$\tan \alpha_0 = \frac{q}{2H}(h \coth r - L). \tag{10.12b}$$

Similarly

$$\tan \alpha_1 = \left.\frac{dy}{dx}\right|_{x=l} = \sinh(2r + a_1) \tag{10.13a}$$

or

$$\tan \alpha_1 = \frac{q}{2H}(h \coth r + L). \tag{10.13b}$$

10.4 Variation of H Due to Changes in l, h, and L

As the variation of stress in a flexible cable is nonlinear with respect to any geometrical change in dimensions, it will simplify the analysis if small changes in such variables as l, h, and L are first treated independently. Thus, if the relationship expressed by equation 10.11*b* is used to

determine the effect on H and r of a small change in l only, the following results are obtained.

$$\sqrt{L^2 - h^2} = \frac{2H}{q} \sinh r \qquad (10.11b)$$

in which

$$r = \frac{ql}{2H}$$

Then

$$(a) \qquad \frac{\partial\sqrt{L^2 - h^2}}{\partial l} = \frac{2}{q}\left(\frac{\partial H}{\partial l} \sinh r + H \cosh r \frac{\partial r}{\partial l}\right)$$

$$(b) \qquad \frac{\partial r}{\partial l} = \frac{q}{2H} - \frac{ql}{2H^2}\frac{\partial H}{\partial l}$$

Combining equations (a) and (b) and noting that

$$\frac{\partial\sqrt{L^2 - h^2}}{\partial l} = 0$$

since L and h are considered as independent variables, we obtain

$$\frac{2}{q} \sinh r \frac{\partial H}{\partial l} + \cosh r - \frac{l}{H}\frac{\partial H}{\partial l} \cosh r = 0$$

from which

$$\frac{\partial H}{\partial l} = \frac{1}{l/H - (2/q) \tanh r} \qquad (10.14a)$$

Similarly, the change in H with respect to L only can be evaluated from equation 10.11b as follows:

$$\frac{\partial\sqrt{L^2 - h^2}}{\partial L} = \frac{L}{\sqrt{L^2 - h^2}} = \frac{2}{q} \sinh r \frac{\partial H}{\partial L} - \frac{l}{H} \cosh r \frac{\partial H}{\partial L}$$

which gives

$$\frac{\partial H}{\partial L} = \frac{L}{\sqrt{L^2 - h^2}\left[\frac{2}{q} \sinh r - (l/H) \cosh r\right]} \qquad (10.14b)$$

Likewise the effect upon H of a variation in h is found to be

$$\frac{\partial\sqrt{L^2 - h^2}}{\partial h} = \frac{-h}{\sqrt{L^2 - h^2}} = \frac{2}{q} \sinh r \frac{\partial H}{\partial h} - \frac{l}{H} \cosh r \frac{\partial H}{\partial h}$$

or

$$\frac{\partial H}{\partial h} = \frac{-h}{\sqrt{L^2 - h^2}\left[\frac{2}{q} \sinh r - (l/H) \cosh r\right]} \qquad (10.14c)$$

Equations 10.14*a*, *b*, *c* can be simplified by replacing the hyperbolic terms with the following series since most practical values of r are considerably less than unity.

$$\sinh r \simeq r\left(1 + \frac{r^2}{6} + \frac{r^4}{120} + \ldots\right)$$

$$\cosh r \simeq 1 + \frac{r^2}{2} + \frac{r^4}{24} + \ldots$$

$$\tanh r \simeq r\left(1 - \frac{r^2}{3} + \frac{2r^4}{15} + \ldots\right)$$

The modified form of the equations are as follows:

$$\frac{\partial H}{\partial l} = \frac{12H^3}{q^2l^3(1 - 0.4r^2)} \cong \frac{12H^3}{q^2l^3}(1 + 0.4r^2) \tag{10.15a}$$

$$\frac{\partial H}{\partial L} = -\frac{12H^3L}{q^2l^3\sqrt{L^2 - h^2}\,(1 + 0.1r^2)} \tag{10.15b}$$

$$\frac{\partial H}{\partial h} = \frac{12H^3h}{q^2l^3\sqrt{L^2 - h^2}\,(1 + 0.1r^2)} \tag{10.15c}$$

In addition to these relations, the following simplified form for determining L can be obtained from equation 10.11*b* by replacing $\sinh r$ by several terms of the corresponding series.

$$L = \bar{L} + \frac{l^2r^2}{6\bar{L}} + \frac{l^2r^4}{45\bar{L}} - \frac{l^4r^4}{72\bar{L}^3} \tag{10.16}$$

10.5 *Resultant Change ΔH Produced by Combined Effects of Δl, Δh, and ΔL*

From the fundamental relation

$$dH = \frac{\partial H}{\partial l}\,dl + \frac{\partial H}{\partial L}\,dL + \frac{\partial H}{\partial h}\,dh \tag{10.17}$$

and from the value of the partial derivatives given by equations 10.15*a*, *b*, *c* the following integral equation can be established.

$$\int_{H_0}^{H}\frac{dH}{H^3} = \frac{12}{q^2}\int_{l_0}^{l}\frac{1 + 0.4r^2}{l^3}\,dl - \frac{12}{q^2l^3}\int_{L_0}^{L}\frac{L\,dL}{\sqrt{L^2 - h^2}(1 + 0.1r^2)} + \frac{12}{q^2l^3}\int_{h_0}^{h}\frac{h\,dh}{\sqrt{L^2 - h^2}(1 + 0.1r^2)} \tag{10.18}$$

If, during the integration indicated, the following substitutions are made

$$\Delta l = l - l_0, \qquad \Delta L = L - L_0, \qquad \Delta h = h - h_0$$

and a constant value of r equal to r_i, a mean value, is used, the following relations as given by equation 10.19 are obtained.

$$\frac{H^2 - H_0^2}{H^2 H_0^2} = \frac{24}{q^2 l^3}\left[(1 + 0.4r_i^2)\,\Delta l - \frac{L\,\Delta L - h\,\Delta h}{\sqrt{L^2 - h^2}(1 + 0.1r_i^2)}\right] \quad (10.19)$$

By equations 10.9*c* and 10.11*b*, equation 10.19 can be changed to the form

$$\frac{H^2 - H_0^2}{H^2} = \frac{6}{r_0^2 l}\left[(1 + 0.4r_i^2)\,\Delta l - \frac{L\,\Delta L - h\,\Delta h}{l(1 + \frac{1}{6}r_0^2)(1 + 0.1r_i^2)}\right] = K \quad (10.20a)$$

from which

$$H^2 - H_0^2 = KH^2$$

or

$$H = \frac{H_0}{\sqrt{1 - K}} \quad (10.20b)$$

If K is considerably less than unity, equation 10.20*b* may be written in the form

$$H - H_0 = \Delta H = H_0[(1 - K)^{-\frac{1}{2}} - 1]$$

and since

$$(1 - K)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}K + \tfrac{3}{8}K^2 + \tfrac{5}{16}K^3 + \tfrac{35}{128}K^4 + \dots$$

then

$$\Delta H = \frac{H_0 K}{2}\left(1 + \frac{3}{4}K + \frac{5}{8}K^2 + \frac{35}{64}K^3 + \dots\right) \quad (10.20c)$$

10.6 Structural Use of Cables

High strength steel wires and cables have been used in many types of structures such as suspension bridges, long-span roofs, guyed towers, prestressed concrete members, rigid airships, tramway lifts, and construction conveyors. The basic equations of equilibrium for wires and cables when subjected only to tensile stress have been treated in the preceding articles for certain loading conditions. However, since the change in the coordinate system defining the position of the members must be considered and since the initial stress is arbitrary, then any analysis involves both a nonlinear and statically indeterminate problem. When the quantities T or H are constant for any value of x, the solution for vertical or radial loading is mathematically feasible as illustrated in Example 10.1.

Various types of problems are encountered in design and construction which involve the determination of the stresses and displacements due not only to variations in loading and initial stress but also to temperature

changes and movement of the supports. In considering changes in length of a cable because of stress, we can use the general form

$$\Delta L = \frac{1}{A_c E_c} \int_L (T - T_0)\, ds + \alpha L t \qquad (10.21a)$$

where A_c = area of cable
E_c = modulus of elasticity of cable
α = coefficient of linear expansion
t = change in temperature

Since

$$T = H \cosh \left(\frac{q}{H} x + a_1 \right)$$

and

$$\frac{ds}{dx}\, dx = \cosh \left(\frac{q}{H} x + a_1 \right) dx$$

equation 10.21a can be written in the form

$$\Delta L = \frac{1}{A_c E_c} \int_0^l \left[H \cosh^2 \left(\frac{q}{H} x + a_1 \right) - H_0 \cosh^2 \left(\frac{qx}{H_0} + a_1 \right) \right] dx + \alpha L t$$

An approximate solution of this integral was presented by Donald Dean in the form (Ref. 16)

$$\Delta L \cong \frac{L_0^2 \Delta H}{l A_c E_c} \left[1 - \frac{r_i^2}{6} \left(4 - \frac{l^2}{L_0^2} \right) \right] + \alpha L t \qquad (10.21b)$$

in which r_i is determined from the mean value of H, or $H_o + \Delta H/2$. This value of ΔL should be used in equation 10.20a.

Because a tower also undergoes a change in height Δh resulting from any variation in the vertical components of the cable stress, this effect may have an influence on equation 10.20a. Actually this effect is small and can often be neglected, particularly in the preliminary analysis.

To determine the value of Δh in terms of ΔH and the elastic properties of a tower, Dean recommends the following procedure.

$$V = \frac{q}{2} (h \coth r + L)$$

$$r = \frac{ql}{2H}$$

$$dV = \frac{h}{l} r^2 \operatorname{csch}^2 r\, dH$$

$$\operatorname{csch}^2 r = \frac{1}{r^2} - \frac{1}{3} + \frac{r^2}{36}$$

Using the first two terms of the series for $\operatorname{csch}^2 r$ gives the change in the vertical force in any cable as

$$\Delta V = \frac{h}{l}\left(1 - \frac{1}{3} r_1^2\right)\Delta H \tag{10.21c}$$

where r_i again represents the mean value of r and must be estimated. Assuming that the total change in V for all cables is represented by $\Sigma \Delta V$ and the effective area of the tower resisting this force by $\bar{A}_t$,

$$\Delta h = \frac{h \Sigma \Delta V}{\bar{A}_t E_t} + \alpha h t \tag{10.21d}$$

Equations 10.21c and d give the relation between Δh and ΔH to use in equation 10.20a.

Since Δl is not directly dependent on the stress in the cable, although of course related to it, it will be treated hereafter as one of the independent variables. Several examples will be given to illustrate the use of the preceding equations.

Example 10.2 The tower shown in Fig. 10.5 is supported by eight cables $C1$ and $C2$ which are placed during a temperature of 80°F. The area A_c of each cable is 1.4 in.^{2}, E_c is 24×10^6 lb per square inch, and q is 5.2 lb per foot.

Determine the tension T at both ends of cable $C1$, the vertical component V_{ca}, the maximum sag s, and the total length L if the value of

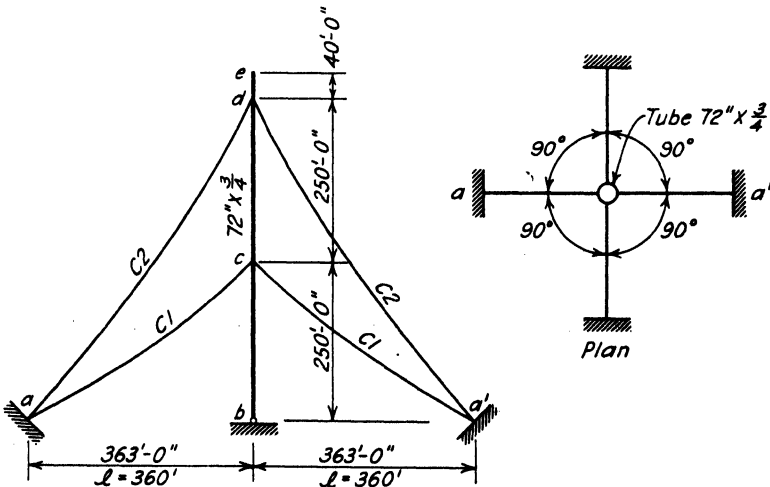



Fig. 10.5

H_o is 40,000 lb.

$$r_0 = \frac{(5.2)(360)}{(2)(40{,}000)} = 0.02340$$

$$a_1 = \sinh^{-1} \frac{(5.2)(250)}{(2)(40{,}000)(0.0234)} - 0.02340$$

$$a_1 = 0.6481 - 0.0234 = 0.6247$$

$$T_{ac} = H \cosh a_1 = (40{,}000)(1.20157) = 48{,}063 \text{ lb}$$

$$T_{ca} = H \cosh (2r + a_1) = 40{,}000 \cosh 0.6715$$

$$= (40{,}000)(1.2341) = 49{,}364 \text{ lb}$$

$$V_{ca} = H \tan \alpha_1 = (H) \frac{q}{2H} [h \coth r + L]$$

or

$$V_{ca} = \frac{q}{2} [h \coth r + L]$$

$$\coth r = \frac{1}{r} + \frac{r}{3} - \frac{r^3}{45} + \frac{2r^5}{945} + \ldots = \frac{1}{0.0234} + \frac{0.0234}{3} = 42.735 + 0.008 = 42.743$$

$$\bar{L} = \sqrt{(360)^2 + (250)^2} = 438.29$$

From equation 10.16

$$L = \bar{L} + \frac{l^2 r^2}{6\bar{L}} + \frac{l^2 r^4}{45\bar{L}} - \frac{l^4 r^4}{72\bar{L}^3}$$

$$\frac{l^2 r^2}{\bar{L}} = \frac{(360)^2(0.0234)^2}{438.29} = 0.162$$

$$L = 438.29 + 0.027 = 438.317 \text{ ft}$$

$$V_{ca} = \frac{5.2}{2} [(250)(42.743) + 438.32] = 28{,}923 \text{ lb}$$

The maximum sag s (Fig. 10.4) will occur where the slope of the cable is h/l or from equation 10.8*b*

$$\frac{h}{l} = \sinh \left(\frac{q}{H} x_s + a_1\right)$$

$$x_s = \frac{H}{q}\left(\sinh^{-1} \frac{h}{l} - a_1\right)$$

or

$$x_s = \frac{40{,}000}{5.2}\left(\sinh^{-1}\frac{250}{360} - a_1\right)$$

$$= \frac{40{,}000}{5.2}(0.6481 - 0.6247) = 180.00 \text{ ft}$$

$$y_s = \frac{H}{q}\left[\cosh\left(\sinh^{-1}\frac{h}{l}\right) - \cosh a_1\right]$$

However,

$$\cosh\left(\sinh^{-1}\frac{h}{l}\right) = \sqrt{1 + (h/l)^2} = \frac{L}{l} = \sec\bar{\alpha}$$

$$y_s = \frac{H}{q}(\sec\bar{\alpha} - \cosh a_1)$$

$$y_s = \frac{40{,}000}{5.2}(1.21747 - 1.20157) = 122.3 \text{ ft}$$

$$s = x_s \sin\bar{\alpha} - y_s \cos\bar{\alpha}$$

$$s = (180.00)(0.5704) - (122.30)(0.8214)$$

giving

$$s = 102.67 - 100.46 = 2.21 \text{ ft}$$

Example 10.3 The value of H for the cable $C1$ in Example 10.2 will be calculated for a drop in temperature of 50°F in both cable and tower.

Using equation 10.21*b* with trial value of ΔH assumed as 6000 lb gives

$$r_i = \frac{(5.2)(360)}{(2)(40{,}000 + 3000)} = \frac{1872}{86{,}000} = 0.02177$$

$$r_i^2 = 0.000474 \qquad \frac{l}{L_0} = \frac{360.0}{438.3} = 0.8214$$

$$\Delta L = \frac{(438.32)^2}{(360)(1.4)(24)(10^6)}\left[1 - \frac{0.000474}{6}(4 - 0.6747)\right]\Delta H$$

$$+ (6.7)(10^{-6})(438.3)(-50)$$

or

$$\Delta L = 0.00001588(1 - 0.0002627)\,\Delta H - 0.1468$$

$$\Delta L = 0.000015876\,\Delta H - 0.1468$$

From equation 10.21c, the change in the vertical component at the tower for one cable is

$$\Delta V = \frac{250}{260}\left(1 - \frac{0.000474}{3}\right)\Delta H$$

or

$$\Delta V = (0.6944)(0.99974\,\Delta H) = 0.69422\,\Delta H$$

If ΔV is assumed to be resisted by one-fourth the area of the tower,

$$\overline{A}_t = 72 \times \pi \times 0.75 = 169.7 \text{ in.}^2$$

$$\Delta h = -\frac{0.69422\,\Delta H}{(169.7/4)(29)(10^6)}(250) + (6.7)(10^{-6})(250)(-50)$$

or

$$\Delta h = -0.1411 \times 10^{-6}\,\Delta H - 0.08375$$

Assuming Δl equal to zero, equation 10.20a gives

$$K = \frac{6}{r_0^{\,2}l^2(1 + r_0^{\,2}/6)(1 + 0.1r_i^{\,2})}(-L\,\Delta L + h\,\Delta h)$$

or

$$K = \frac{6}{(70.964)(1.00009)(1.00005)}[-438.32(0.000015876\,\Delta H - 0.1468)]$$

$$+ 250(-0.1411 \times 10^{-6}\,\Delta H - 0.08375)$$

$$K = (0.08454)(-0.006959\,\Delta H + 64.345 - 35.28 \times 10^{-6}\,\Delta H - 20.938)$$

$$K = 0.08454(-0.006924\,\Delta H + 43.407)$$

giving

$$K = -0.000585\,\Delta H + 3.67$$

In addition,

$$\Delta H = \frac{H_0}{\sqrt{1 - K}} - H_0 = \frac{40{,}000}{\sqrt{1 - K}} - 40{,}000$$

By trial it can be shown that a value of ΔH equal to 5860 lb or an H value of 45,860 lb will satisfy these equations.

Example 10.4 The preceding equations will be used to obtain the value of H in cable $C1$ if point c moves horizontally so that Δl is 0.2 ft. Neglect the effect of any change in Δh and assume no temperature change.

$$K = \frac{6}{r_0^{\,2}l}\left\{(1 + 0.4r_i^{\,2})\,\Delta l - \frac{L^3[1 - (r_i^{\,2}/6)(4 - l^2/L^2)]\,\Delta H}{l^2 A_c E_c(1 + r_0^{\,2}/6)(1 + 0.1r_i^{\,2})}\right\}$$

If ΔH is assumed to be 9700 lb

$$r_i = \frac{(5.2)(360)}{2(40{,}000 + 4850)} = \frac{1872}{89{,}700} = 0.02087$$

$$r_i^2 = (0.02087)^2 = 0.0004356 \qquad r_0^2 = 0.0005476$$

$$A_cE_c = (1.4)(24)(10^6) = 33.6 \times 10^6 \text{ lb}$$

$$\frac{L^3}{l^2A_cE_c} = \frac{(438.32)^3}{(360)^2(33.6)(10^6)} = \frac{84{,}211{,}967}{(129{,}600)(33.6)(10^6)}$$

$$= \frac{649.837}{(33.6)(10^6)} = 19.34 \times 10^{-6}$$

$$1 + \frac{r_0^2}{6} = 1 + \frac{0.0005476}{6} = 1.00009$$

$$\frac{6}{r_0^2 l} = \frac{6}{(0.0234)^2(360)} = \frac{6}{0.19714} = 30.435$$

$$K = 30.435\left\{[1 + 0.4r_i^2]\Delta l - \frac{(19.34)(10^{-6})}{1.00009}\left[1 - \frac{r_i^2}{6}(4 - 0.674)\right]\Delta H\right\}$$

$$K = 30.435(1 + 0.4r_i^2)\,\Delta l - (588.61)(10^{-6})(1 - 0.5551r_i^2)\,\Delta H$$

or

$$K = 30.488\,\Delta l - (0.58846)(10^{-3})\,\Delta H$$

which for Δl equal to 0.2 ft gives

$$K = 6.193 - 0.0005885\,\Delta H$$

Also

$$H = \frac{H_0}{\sqrt{1 - K}}$$

A solution by trial gives ΔH about 9760 lb or $H = 49{,}760$ lb. These equations are very sensitive with respect to changes in ΔH.

10.7 Compression Members with Transverse Loads

When a flexible member with some flexural strength is subjected to combined axial and transverse loads as in Fig. 10.6*a*, the bending moments caused by the axial load P acting through the displacement y may be of considerable magnitude. Since the calculation of the stresses in such members is fully discussed in many books on strength of materials, only the deformation equations that are necessary for the analysis of continuous beams and frames are treated here. In the following derivation it is assumed that the stresses are within the proportional limit and that the member has sufficient rigidity to prevent buckling.

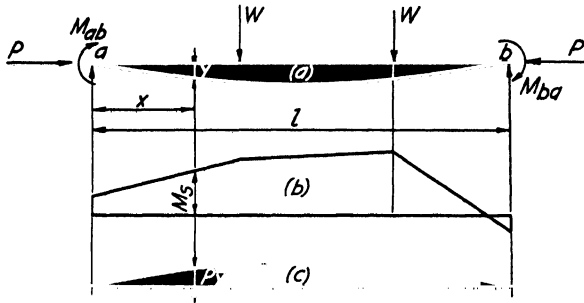



Fig. 10.6

The bending moment M_x at any section of the member ab (Fig. 10.6), is equal to

$$M_x = M_s + Py \tag{10.22}$$

where M_s is the bending moment when the axial load P is not acting (Fig. 10.6b). The equation of the elastic curve is

$$\frac{d^2y}{dx^2} = -\frac{M_x}{EI} \tag{10.23}$$

and, for a member with constant EI, this equation becomes

$$\frac{d^2y}{dx^2} + a^2y = -a^2\left(\frac{M_s}{P}\right) \tag{10.24}$$

in which $a^2 = P/EI$. The general solution of this equation when M_s is a polynomial is

$$y = A\cos ax + B\sin ax - \frac{M_s}{P} + \frac{1}{a^2}\frac{M_s''}{P} - \frac{1}{a^4}\frac{M_s^{\text{iv}}}{P} + \cdots \tag{10.25}$$

in which

$$M_s'' = \frac{d^2M_s}{dx^2} \qquad M^{\text{iv}} = \frac{d^4M_s}{dx^4}$$

For a uniform load of w lb per unit length and end couples of M_{ab} and M_{ba},

$$M_s = \frac{M_{ab}(l-x)}{l} - M_{ba}\frac{x}{l} + \frac{wl}{2}x - \frac{wx^2}{2}$$

$$M_s'' = -w, \qquad M_s^{\text{iv}} = 0$$

or

$$y = A\cos ax + B\sin ax - \frac{M_{ab}(l-x)}{Pl} + \frac{M_{ba}x}{Pl} - \frac{wl}{2}\frac{x}{P} + \frac{wx^2}{2P} - \frac{w}{a^2P} \tag{10.26}$$

The boundary conditions are

$$y = 0 \qquad x = 0 \qquad \text{or} \qquad A = \frac{M_{ab}}{P} + \frac{w}{a^2P} \tag{10.27a}$$

$$y = 0 \qquad x = l$$

or

$$B = \frac{-(M_{ab}/P + w/a^2P)\cos al - M_{ba}/P + w/a^2P}{\sin al} \tag{10.27b}$$

Since the slope at any point of the elastic curve can be obtained from the equation

$$\frac{dy}{dx} = -Aa \sin ax + Ba \cos ax - \frac{M_s'}{P} + \frac{1}{a^2}\frac{M_s'''}{P} \tag{10.28a}$$

the end rotations θ_a and θ_b can be determined by substituting the proper value of x, that is, for $x = 0$

$$\theta_a = \left(\frac{dy}{dx}\right)_{x=0} = Ba - \frac{M_s'}{P} \tag{10.28b}$$

After substituting the value of B and M_s' for $x = 0$ in equation 10.28b and replacing al by α, we obtain the following relation:

$$\theta_a = \frac{M_{ab}}{Pl}(1 - \alpha \cot \alpha) - \frac{M_{ba}}{Pl}(\alpha \operatorname{cosec} \alpha - 1) + \frac{wl}{P}\left(\frac{\tan \alpha/2}{\alpha} - \frac{1}{2}\right) \tag{10.29a}$$

In the same manner, for $x = l$,

$$\theta_b = \left(\frac{dy}{dx}\right)_{x=l} = -\frac{M_{ab}}{Pl}(\alpha \operatorname{cosec} \alpha - 1) + \frac{M_{ba}}{Pl}(1 - \alpha \cot \alpha) - \frac{wl}{P}\left(\frac{\tan \alpha/2}{\alpha} - \frac{1}{2}\right) \tag{10.29b}$$

in which $\alpha = \sqrt{Pl^2/EI}$. Solving equation 10.29a and b for M_{ab} and M_{ba} gives the usual form of the slope deflection equations

$$M_{ab} = C_1K\theta_a + C_2K\theta_b + (C_1 + C_2)K\frac{\Delta}{L} + M_{Fab} \tag{10.30a}$$

$$M_{ba} = C_2K\theta_a + C_1K\theta_b + (C_1 + C_2)K\frac{\Delta}{L} + M_{Fba} \tag{10.30b}$$

in which

$$C_1 = \frac{1 - \alpha \cot \alpha}{\dfrac{2 \tan \alpha/2}{\alpha} - 1} \tag{10.31a}$$

$$C_2 = \frac{\alpha \operatorname{cosec} \alpha - 1}{\dfrac{2 \tan \alpha/2}{\alpha} - 1} \tag{10.31b}$$

$$M_{Fab} = -M_{Fba} = \left(\frac{1 - \dfrac{2 \tan \alpha/2}{\alpha}}{2\alpha \tan \alpha/2} \right) wl^2 \tag{10.31c}$$

$$K = \frac{EI}{l} \qquad \alpha = \sqrt{Pl^2/EI}$$

By replacing the trigonometric functions of α by their corresponding series, Professor Donald Dean transformed equations 10.31a, b, c into the following form:

$$C_1 = 4\left(1 - \frac{\alpha^2}{30} - \frac{11\alpha^4}{25{,}200} - \frac{7\alpha^6}{756{,}000} \cdots\right) \tag{10.32a}$$

$$C_2 = 2\left(1 + \frac{\alpha^2}{60} + \frac{13\alpha^4}{25{,}200} + \frac{11\alpha^6}{756{,}000} + \cdots\right) \tag{10.32b}$$

$$M_{Fab} = -M_{Fba} = -\frac{wl^2}{12}\left(1 + \frac{\alpha^2}{60} + \frac{\alpha^4}{2520} + \frac{\alpha^6}{100{,}800} + \cdots\right) \tag{10.32c}$$

After the values of C_1K, C_2K, and the fixed-end moments are determined, the distribution and carry-over factors are calculated in the usual manner. The carry-over factor is, of course,

$$\frac{C_2}{C_1} = \frac{\alpha \operatorname{cosec} \alpha - 1}{1 - \alpha \cot \alpha} \tag{10.31d}$$

A diagram in the Appendix gives C_1 and C_2/C_1 values for various values of α.

10.8 Tension Members with Transverse Loads

If an axial tensile load instead of compression is used, the equations become

$$M_x = M_s - Py \tag{10.33a}$$

$$\frac{d^2y}{dx^2} - a^2y = -a^2\left(\frac{M_s}{P}\right) \tag{10.33b}$$

in which $a^2 = P/EI$.

The general solution of this differential equation gives the equation of the elastic curve as

$$y = A \cosh ax + B \sinh ax + \frac{M_s}{P} + \frac{1}{a^2}\frac{M_s''}{P} + \frac{1}{a^4}\frac{M_s^{iv}}{P} + \cdots \qquad (10.33c)$$

If the evaluation of the constants A and B and the end slopes θ_a and θ_b is carried out for a uniform distribution w pounds per unit length and for end couples M_{ab} and M_{ba} in the same manner as was previously used for the axial compressive force, we again obtain

$$M_{ab} = C_1 K \theta_a + C_2 K \theta_b + (C_1 + C_2) K \frac{\Delta}{L} + M_{Fab} \qquad (10.30a)$$

$$M_{ba} = C_2 K \theta_a + C_1 K \theta_b + (C_1 + C_2) K \frac{\Delta}{L} + M_{Fba} \qquad (10.30b)$$

in which

$$C_1 = \frac{\alpha \coth \alpha - 1}{1 - \dfrac{2 \tanh \alpha/2}{\alpha}} \qquad (10.34a)$$

$$C_2 = \frac{1 - \alpha \operatorname{cosch} \alpha}{1 - \dfrac{2 \tanh \alpha/2}{\alpha}} \qquad (10.34b)$$

$$M_{Fab} = -M_{Fba} = \left(\frac{2 \tanh \alpha/2 - \alpha}{2\alpha^2 \tanh \alpha/2}\right) wl^2 \qquad (10.34c)$$

$$\alpha = \sqrt{Pl^2/EI}$$

$$K = \frac{EI}{l}$$

Values of C_1 and C_2/C_1 for various values of α can be obtained from diagrams in the Appendix.

10.9 Principle of Superposition—Reciprocal Theorem

From equations 10.29*a*, *b* it is apparent that the displacements y and θ are linear functions of M_{ab}, M_{ba}, and w if P, E, I, and l are kept constant. Consequently, the displacements due to any combination of transverse loads can be added algebraically in any order provided all displacements are calculated with the same value of the axial load P acting. However, if the axial load varies with the transverse loading, as it frequently does, the true value of P may be difficult to obtain. Fortunately, the value of the axial load P in most structures is not sensitive to changes in the stiffness of the member.

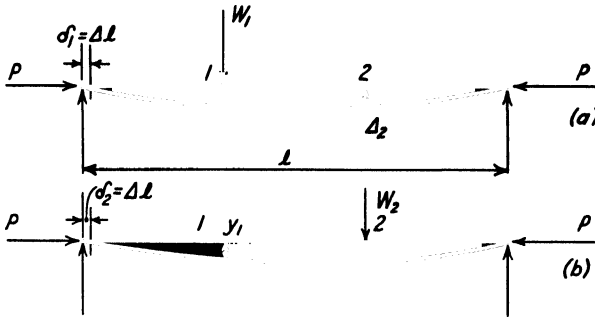



Fig. 10.7

If the development of the reciprocal theorem in Article 2.5 is studied, it will be found that this discussion will also apply to members with both transverse and axial loads if the axial load P and the value of α are kept constant for all systems of applied loads. Thus, if we consider the two force systems defined by W_1 (Fig. 10.7*a*) and W_2 (Fig. 10.7*b*), the reciprocal theorem will give

$$W_1 y_1 + P\delta_2 = W_2 \Delta_2 + P\delta_1$$

but, since δ_2 is equal to δ_1

$$P(\delta_2 - \delta_1) = 0$$

if only the changes in length due to the axial loads are considered.

Example 10.5 The fixed-end moments for the member ab in Fig. 10.8 will be calculated by the reciprocal theorem for a value of α equal to two. If any moment M_{ab} is applied to the member as shown in Fig. 10.8, producing the elastic curve shown, then, by the reciprocal theorem,

$$M_{Fab}\theta_a + Wy = 0$$

or

$$M_{Fab} = -W\left(\frac{y}{\theta_a}\right)$$

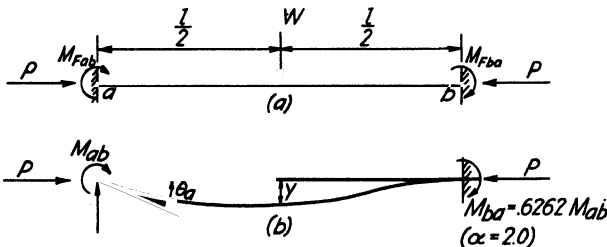



Fig. 10.8

From equation 10.31*d*, the value of M_{ba} is

$$M_{ba} = \frac{C_2}{C_1} M_{ab} = \frac{\alpha \operatorname{cosec} \alpha - 1}{1 - \alpha \cot \alpha} M_{ab} = 0.6262 M_{ab}$$

From equations 10.26 and 10.27

$$(y)_{x=l/2} = \frac{M_{ab}}{P} \cos \frac{\alpha}{2} - \frac{M_{ab}}{P} \cot \alpha \sin \frac{\alpha}{2} - \frac{M_{ba}}{P} \operatorname{cosec} \alpha \sin \frac{\alpha}{2} - \frac{M_{ab}}{2P} + \frac{M_{ba}}{2P}$$

which gives for $\alpha = 2$

$$(y)_{x=l/2} = 0.159 \frac{M_{ab}}{P}$$

From equation 10.29*a*

$$\theta_a = \frac{M_{ab}}{P} (1 - \alpha \cot \alpha) - \frac{M_{ba}}{Pl} (\alpha \operatorname{cosec} \alpha - 1)$$

from which

$$\theta_a = 1.1642 \frac{M_{ab}}{Pl}$$

Therefore

$$M_{Fab} = -W \left(\frac{y}{\theta_a} \right) = -W \left(\frac{0.159(M_{ab}/P)}{1.1642(M_{ab}/Pl)} \right)$$

$$= -0.1365\, Wl \qquad \text{(counterclockwise)}$$

The fixed-end moments can be calculated by this method for any position of W and for any value of α.

10.10 Use of Trigonometric Series

The mathematical difficulties involved in the solution of equation 10.24 may often be avoided by the use of trigonometric series in calculating the displacements y and θ. In equations 10.22 and 10.23, the bending moment M_s and displacement y can usually be represented by a Fourier series of the type

$$M_s = \sum_{n=1}^{\infty} A_n' \sin \frac{n\pi x}{l} \tag{10.35a}$$

$$y = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \tag{10.35b}$$

and from equation 10.35*b* we obtain

$$\frac{d^2 y}{dx^2} = -\frac{\pi^2}{l^2} \sum_{n=1}^{\infty} n^2 A_n \, sin \frac{n\pi x}{l}$$

Substituting these values in equation 10.23 gives

$$-EI\frac{\pi^2}{l^2}\sum_{n=1}^{\infty}n^2A_n\sin\frac{n\pi x}{l} = -\sum_{n=1}^{\infty}A_n'\sin\frac{n\pi x}{l} - P\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l} \quad (10.36a)$$

To satisfy equation 10.36*a* for each term of the series, the coefficients must have the relation

$$A_n = \frac{l^2}{EI}\left(\frac{A_n'}{n^2\pi^2 - \alpha^2}\right) \quad (10.36b)$$

The elastic curve is therefore known whenever the coefficients A_n' for the bending moments M_s are determined. These coefficients can ordinarily be calculated from the requirement that

$$A_n' = \frac{2}{l}\int_0^l M_s \sin\frac{n\pi x}{l}\,dx \quad (10.36c)$$

For example, if a concentrated load W is applied at a distance c from one end, the equations for M_s are

$$M_s = \frac{W(l-c)}{l}x \qquad \text{for } x \to 0 \text{ to } c$$

$$M_s = \frac{Wc}{l}(l-x) \qquad \text{for } x \to c \text{ to } l$$

From equation 10.36*c*, we obtain

$$A_n' = \frac{2}{l}\left[\int_0^c \frac{W(l-c)}{l}x\sin\frac{n\pi x}{l}\,dx + \int_c^l \frac{Wc(l-x)}{l}\sin\frac{n\pi x}{l}\,dx\right]$$

from which

$$A_n' = \frac{2Wl}{n^2\pi^2}\sin\frac{n\pi c}{l} \quad (10.37a)$$

and the equation of the elastic curve as given by equations 10.35*b* and 10.36*b* is

$$A_n = \frac{2Wl^3}{\pi^2 EI}\left[\frac{\sin(n\pi c/l)}{n^2(n^2\pi^2 - \alpha^2)}\right] \quad (10.37b)$$

$$y = \frac{2Wl^3}{\pi^2 EI}\sum_{n=1}^{\infty}\frac{\sin(n\pi c/l)\sin(n\pi x/l)}{n^2(n^2\pi^2 - \alpha^2)} \quad (10.37c)$$

$$\frac{dy}{dx} = \frac{2Wl^2}{\pi EI}\sum_{n=1}^{\infty}\frac{\sin(n\pi c/l)\cos(n\pi x/l)}{n(n^2\pi^2 - \alpha^2)} \quad (10.37d)$$

The end slopes are readily obtained from equation 10.37*d*, and, with the end rotations known, the fixed-end moments can be calculated from

equations 10.30a and b in the same manner as for beams without axial loads. By changing the term $n^2\pi^2 - \alpha^2$ in the denominator of equation 10.36b to $n^2\pi^2 + \alpha^2$, equations 10.37b, c, d can also be used when an axial tensile force is acting.

Example 10.6 The fixed-end moments for the member in Fig. 10.8a will be calculated from equations 10.30a and b by taking θ_a and θ_b equal to the rotations due to the transverse load W but with opposite sign.

From equation 10.37d, for

$$x = 0 \qquad c = \frac{l}{2}$$

$$\theta_a = -\theta_b = \frac{2Wl^2}{\pi EI}\left[\frac{1}{\pi^2 - \alpha^2} - \frac{1}{3(9\pi^2 - \alpha^2)} + \frac{1}{5(25\pi^2 - \alpha^2)} \cdots\right]$$

For

$$\alpha = 2 \qquad \frac{EI}{l} = K$$

$$\theta_a = -\theta_b = \frac{2Wl}{\pi K}(0.1703 - 0.0039) = 0.1059\frac{Wl}{K}$$

Using equations 10.31a and b, we obtain

$$C_1 = \frac{1 - 2\cot 2}{(2\tan 1/2) - 1} = \frac{1.9153}{0.55741} = 3.435$$

$$C_2 = \frac{2\operatorname{cosec} 2 - 1}{(2\tan 1/2) - 1} = \frac{1.1995}{0.55741} = 2.15$$

and from equation 10.30a,

$$M_{Fab} = 3.435K\left(-0.1059\frac{Wl}{K}\right) + 2.15K\left(0.1059\frac{Wl}{K}\right) = -0.1361Wl$$

which agrees closely with the result in Example 10.5.

10.11 Continuous Beams with Axial Loads

After the coefficients and fixed-end moments for each span of a continuous beam with axial loads have been calculated by equation 10.31 or 10.34, the actual end moments can then be obtained by the moment-distribution method in the usual manner. For axial compressive loads the carry-over factors are larger than 0.5, and as these carry-over factors increase the rate of convergence of the corrections decreases. When the value of α is equal to π, the carry-over factor is equal to unity. For this value the convergence by the moment-distribution method is likely to

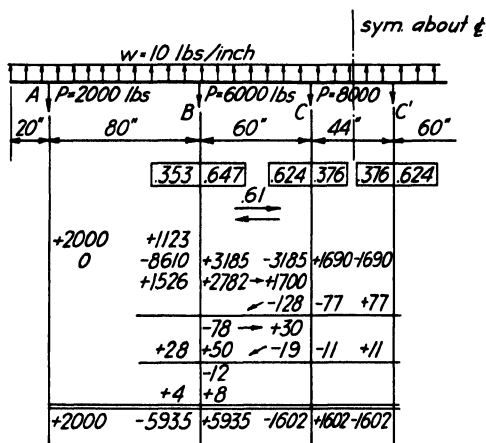



Fig. 10.9

be slow and therefore for such problems the direct use of equations 10.30*a* and *b* as in the slope deflection method in Chapter 4 is preferable. Both the moment-distribution and slope deflection methods are illustrated in Example 10.7.

Example 10.7 The end moments for the continuous beam shown in Fig. 10.9 will be calculated by both the moment-distribution and slope deflection methods.

For all spans,

$$E = 1.5 \times 10^6 \text{ lb per in.}^2$$

$$I = 4.0 \text{ in.}^4$$

$$EI = 6.0 \times 10^6 \text{ in.}^2\text{-lb}$$

For span AB,

$$\alpha = \left[\frac{(2000)(80)(80)}{(6.0)(10^6)}\right]^{1/2} = 1.46$$

From equations 10.31*a* and *b*,

$$C_1 = \frac{1 - 0.163}{1.225 - 1} = 3.72$$

$$C_2 = \frac{(1.46/0.9939) - 1}{0.225} = 2.09$$

$$C_b' = C_1 - \frac{C_2^{\,2}}{C_1} = 2.55 \qquad \text{(for hinge at } A\text{)}$$

$$M_{FAB} = \frac{1 - 1.225}{(2)(1.46)(0.8949)} wl^2 = (-0.0861)(-10)(80)^2 = 5510 \text{ in.-lb}$$

$$M_{FBA} = -5510 \text{ in.-lb}$$

For a hinge at A,

$$M_{FBA} = -5510 - \left(\frac{2.09}{3.72}\right)(5510) = -8610 \text{ in.-lb}$$

For span BC,

$$\alpha = \left[\frac{(6000)(60)(60)}{(6.0)(10^6)}\right]^{1/2} = 1.893$$

$$C_1 = 3.51 \qquad C_2 = 2.14$$

$$M_{FBC} = 3185 \text{ in.-lb} \qquad M_{FCB} = -3185 \text{ in.-lb}$$

Table 10.1 Distribution and Carry-Over Factors

Member	C	K	CK	r	Carry-Over
			Joint B		
BA	2.55	$\frac{4.0}{80}$	0.1275	0.353	0
BC	3.51	$\frac{4.0}{60}$	0.2340	0.647	0.61
			ΣCK = 0.3615	1.000	
			Joint C		
CB	3.51	$\frac{4.0}{60}$	0.2340	0.624	0.61
CC'	1.55	$\frac{4.0}{44}$	0.1410	0.376	
			ΣCK = 0.375	1.000	

For span CC',

$$\alpha = 1.604 \qquad C_1 = 3.63 \qquad C_2 = 2.08$$

$$M_{FCC'} = 1690 \text{ in.-lb}$$

From symmetry it is known that

$$\theta_{C'} = -\theta_C$$

Therefore

$$M_{CC'} = 3.63K\theta_C + 2.08K\theta_{C'} = 1.55K\theta_C$$

If the stiffness factor of span CC' is taken as $1.55K$, the rotation of θ_C' is taken into consideration and no carry-over is necessary.

The distribution of the fixed-end moments is shown in Fig. 10.9. These operations are performed in the usual way.

When the proper coefficients and fixed-end moments are substituted in equations 10.30*a* and *b*, the end moments for the continuous beam in

Fig. 10.9 can be expressed in the following form:

$$M_{BA} = 0.1275\theta_b - 7487$$
$$M_{BC} = 0.234\theta_b + 0.143\theta_c + 3185$$
$$M_{CB} = 0.143\theta_b + 0.234\theta_c - 3185$$
$$M_{CC'} = 0.141\theta_c + 1690$$

When these expressions for end moments are substituted in the equilibrium equations,

$$M_{BA} + M_{BC} = 0$$
$$M_{CB} + M_{CC'} = 0$$

the following equations are obtained:

$$0.362\theta_b + 0.143\theta_c = 4302$$
$$0.143\theta_b + 0.375\theta_c = 1495$$

from which

$$\theta_b = 12{,}130 \qquad \theta_c = -648$$

If these values of θ are substituted back in the slope deflection equations, the numerical values of the end moments become

$$M_{BA} = 5937 \text{ in.-lb} \qquad M_{CB} = -1602 \text{ in.-lb}$$

10.12 Bending Moments in Beams with Axial Loads

The bending moment at any section of a beam that is subjected to both transverse and axial loads is obtained by substituting the proper value of y from equation 10.25 into equation 10.22, or 10.33*c* into 10.33*a*. The displacement y for a beam with uniform load w and an axial compressive load P, both constant across the span, can be expressed by means of equations 10.26 and 10.27 in the form

$$y = \frac{1}{P}\left(A' \cos k\alpha + B' \sin k\alpha - M_s - \frac{wl^2}{\alpha^2}\right) \tag{10.38}$$

in which

$$A' = M_{ab} + \frac{wl^2}{\alpha^2} \tag{10.39a}$$

$$B' = -\frac{A' \cos \alpha + M_{ba} - wl^2/\alpha_2^{\,2}}{\sin \alpha} \tag{10.39b}$$

$$\alpha = \sqrt{Pl^2/EI}$$

$$k = \frac{x}{l}$$

The bending moment M_x at a distance x from the left support is therefore

$$M_x = M_s + Py = A' \cos k\alpha + B' \sin k\alpha - \frac{wl^2}{\alpha^2} \tag{10.40}$$

Since A' and B' are constants, the bending moment can be calculated for any value of $k\alpha$.

The maximum bending moment is obtained when the value of k satisfies the condition

$$\frac{dM_x}{dk} = -\alpha A' \sin k\alpha + \alpha B' \cos k\alpha = 0 \qquad (10.41a)$$

or

$$\tan k\alpha = \frac{B'}{A'} \qquad (10.41b)$$

Example 10.8 The bending moments at several sections in span BC of the continuous beam in Fig. 10.9 will be calculated by equation 10.40.

$$\alpha = 1.893 \qquad \frac{wl^2}{\alpha^2} = \frac{(-10)(60)(60)}{(1.893)^2} = -10{,}040$$

$$\cos \alpha = -0.31665 \qquad M_{BC} = 5935$$

$$\sin \alpha = 0.94853 \qquad M_{CB} = -1602$$

$$A' = 5935 - 10{,}040 = -4105$$

$$B' = -\frac{(-4105)(-0.31665) + (-1602 + 10{,}040)}{0.94853} = -10{,}270$$

Therefore

$$M_x = -4105 \cos k\alpha - 10{,}270 \sin k\alpha + 10{,}040$$

For various values of k, M_x has the following values:

Table 10.2

$k = \frac{x}{l}$	$k\alpha$	M_x
0	0	5935 in.-lb
0.2	0.379	2430
0.4	0.7572	10
0.6	1.1358	−1000
0.8	1.5144	−440
1.0	1.893	1600

For the minimum moment,

$$\tan k\alpha = \frac{-10{,}270}{-4105} = 2.50$$

$$k\alpha = 1.19$$

$$k = \frac{1.19}{1.89} = 0.629$$

$$M_{\min} = (-4105)(0.372) - (10{,}270)(0.928) + 10{,}040 = -1010 \text{ in.-lb}$$

Example 10.9 The shears and bending moments in the guyed tower that was used in Examples 10.2, 10.3, and 10.4 (note change in thickness of

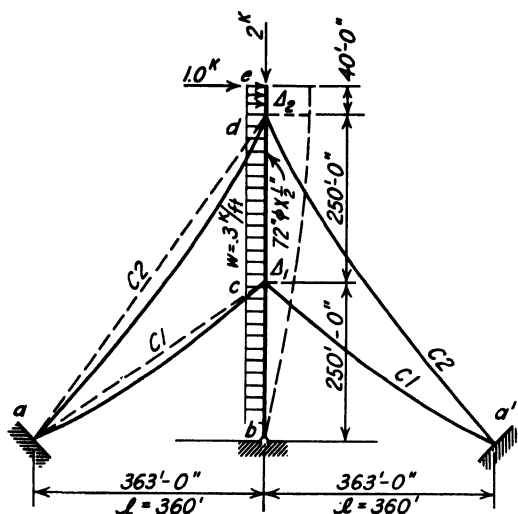



Fig. 10.10a

tube) will be calculated for a dead load on the tower of 430 lb per linear foot and a lateral wind load of 300 lb per linear foot. Concentrated loads of 2 kips vertical and 1 kip horizontal at point e, as shown in Fig. 10.10a, will also be considered. The initial tensions in cables $C1$ and $C2$ give H_0 values of 40 kips and 28 kips respectively. Assuming the weight on the cables as 5.2 lb per linear foot, the tension and vertical components at the tower (see Example 10.2) are

Cables $C1$, $T = 49.36$ kips $V = 28.92$ kips

Cables $C2$, $T = 49.25$ kips $V = 40.5$ kips

If the cables were straight members with the same values of H_0, the corresponding axial forces would be

Cables $C1$, $T = 48.7$ kips $V = 27.8$ kips

Cables $C2$, $T = 48.0$ kips $V = 38.8$ kips

Since the horizontal resistance that the cables will give to the tower shaft at points c and d is an important factor in the analysis, the variation of the horizontal cable force H and the horizontal displacement Δ must be determined. This relationship can be determined from the procedure used in Example 10.4 by assigning different values to Δl, which can be expressed in terms of Δ. If the movement of the shaft is in the same

vertical plane as a cable, Δl is equal to Δ, but will be zero for the cables in a perpendicular plane. However, when the plane of the cable makes an angle ϕ with the vertical plane in which the shaft axis moves, Δl will be equal to $\Delta \cos \phi$.

For the dimensions and initial tensions previously recorded, the variation of H, that is, ΔH, with respect to the horizontal displacement Δ is shown in Fig. 10.10*b* for cables *ca*, *ca'*, *da*, and *da'* that is, for both windward and leeward cables. The effect of the wind load on the cable itself is not included.

The values in Fig. 10.10*b* were calculated by neglecting Δh which gives the value K in the form

$$K = \frac{6}{r_0^2 l}\left\{(1 + 0.4r_i^2)\,\Delta l - \frac{L^3\{1 - (r_i^2/6)[4 - (l^2/L^2)]\}}{l^2 A_c E_c[1 + (r_0^2/6)(1 + 0.1r_i^2)]}\,\Delta H\right\}$$

and

$$H = \frac{H_0}{\sqrt{1 - K}}$$

The values of ΔH and K are negative when Δl, or Δ the horizontal displacement, is negative.

If cables $C1$ and $C2$ are treated as straight members subjected to the horizontal movements Δ_1 and Δ_2 at points c and d respectively, the variation in H can be expressed as

$$\Delta H = \frac{A_c E_c l^2}{L^3}\,\Delta$$

which gives

$$\Delta H_1 = 51.72\Delta_1 \qquad \text{for cable } C1$$

and

$$\Delta H_2 = 18.8\Delta_2 \qquad \text{for cable } C2$$

Δ and H are expressed in terms of feet and kip units. Although this approximation is sufficiently close for member *ca* and *da*, that is, on the windward side, it should not be used for *ca'* and *ca'* in which the relationship of ΔH to Δ is definitely nonlinear. For the first trial solution the following variations will be used (see Fig. 10.10*b*):

For *ca* $\qquad \Delta H = 50.3\Delta_1$

For *ca'* $\qquad \Delta H = -12\text{–}18.6\Delta_1$

For *da* $\qquad \Delta H = 18.0\Delta_2$

For *da'* $\qquad \Delta H = -5.2\text{–}9.4\Delta_2$

The moments and shears in the tower shaft will now be determined by using the methods described in Examples 10.7 and 10.8 and the cable

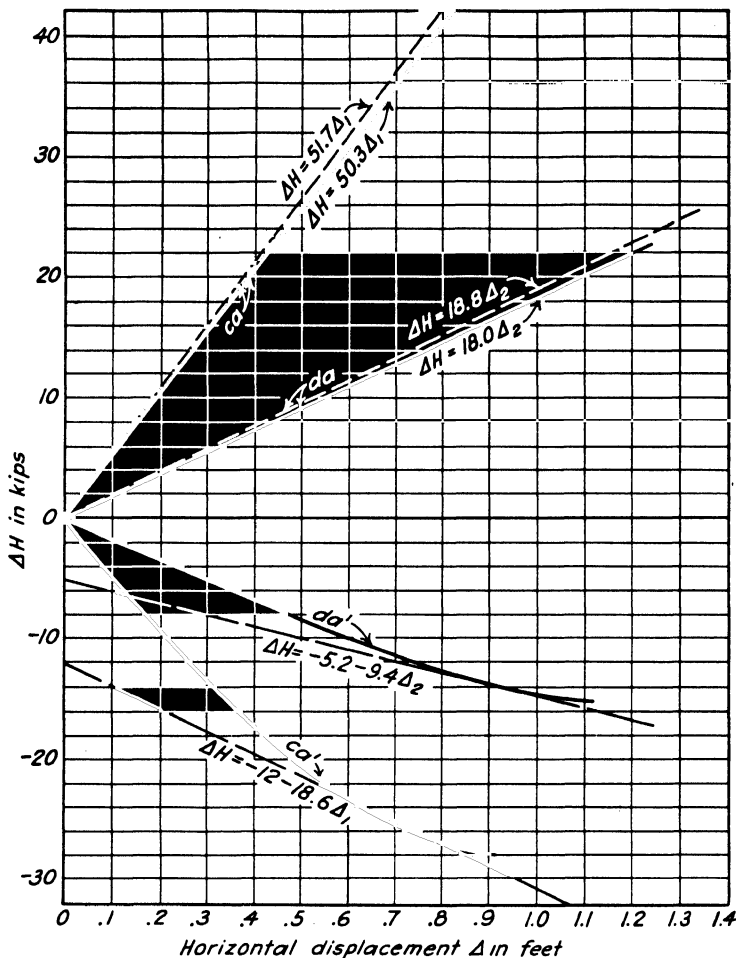



Fig. 10.10b

deformation equations given. The 72 in. by $\frac{1}{2}$ in. tube (total dead load is 0.43 kips per foot) will be treated as a continuous beam column for which the axial loads at the center of the spans are:

For span dc

$$P = (4)(40.5) + (165)(0.43) + 2 = 235.0 \text{ kips}$$

For span cb

$$P = 235.0 + (4)(28.92) + (250)(0.43) = 458.2 \text{ kips}$$

$$I = \pi t r^3 = \pi(0.5)(36)^3 = 73{,}287 \text{ in.}^4$$

For span *bc*

$$\alpha_1 = \left(\frac{(458.2)(250)(250)(144)}{(30{,}000)(73{,}287)}\right)^{\frac{1}{2}} = 1.37, \text{ say } 1.4$$

$$C_1 = 4\left(1 - \frac{1.4^2}{30}\right) = 4(1 - 0.065) = 3.74$$

$$C_2 = 2\left(1 + \frac{1.4^2}{60}\right) = 2.07 \qquad \text{and} \qquad \frac{C_2}{C_1} = 0.555$$

$$M_{Fab} = -\frac{wL^2}{12}\left(1 + \frac{1.4^2}{60}\right) = -0.086wL^2$$

or

$$M_{Fab} = -(0.086)(0.3)(250)^2 = -1615 \text{ ft.-kips}$$

$$C_1' = C_1 - \frac{C_2^{\,2}}{C_1} = 2.59$$

For span *cd*

$$\alpha_2 = \left(\frac{(235)(250)(250)(144)}{(30{,}000)(73{,}287)}\right)^{\frac{1}{2}} = 0.98, \text{ say } 1.0$$

$$C_1 = 4(1 - \tfrac{1}{30}) = 3.87$$

$$C_2 = 2(1 + \tfrac{1}{60}) = 2.03$$

$$\frac{C_2}{C_1} = 0.525 \qquad \text{and} \qquad C_1' = C_1 - \frac{C_2^{\,2}}{C_1} = 2.81$$

$$M_{Fbc} = -\frac{wL^2}{12}\left(1 + \frac{1}{60}\right) = -0.085wL^2$$

or

$$M_{Fbc} = -(0.085)(0.3)(250)^2 = -1595 \text{ ft.-kips}$$

For cantilever *de*

$$M_{de} = -(1)(40) - (0.3)(40)(20) = -280$$

The distribution of the fixed-end moments for the wind load is shown in Fig. 10.10*c*, together with the shears and reactive forces. The moment at point *c* for no translation at *c* and *d* is 2403 ft.-kips and the horizontal reactions necessary to provide this condition are 93.2 kips at *c* and 42.0 kips at *d*.

For any horizontal displacement Δ_1 of point *c* and no displacement of

Table 10.3 Distribution Factors at Joint c

Span	C	K (assumed)	CK	r
cb	2.59	1	2.59	0.48
cd	2.81	1	2.81	0.52
			$\Sigma CK = 5.40$	1.00

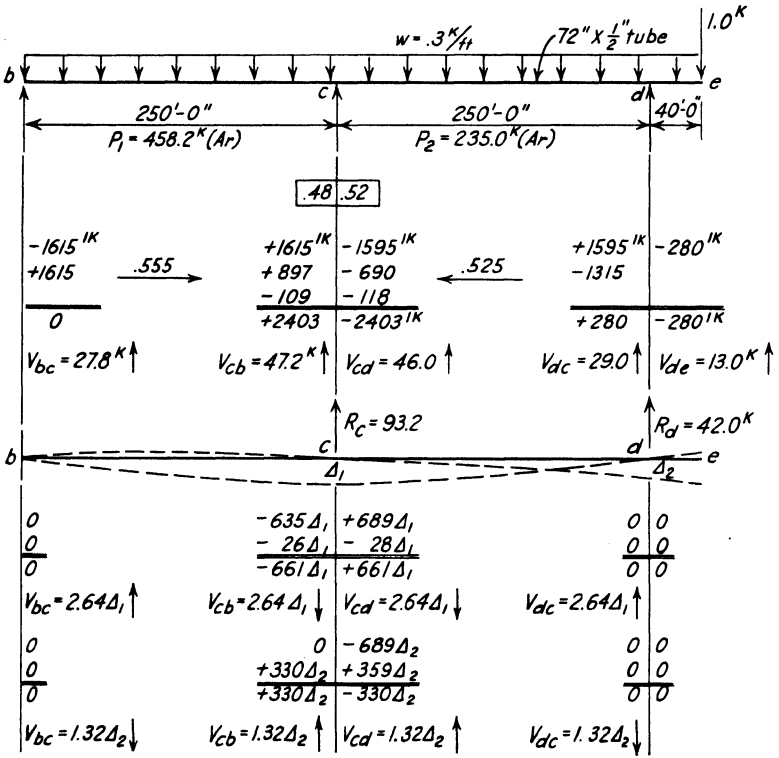



Fig. 10.10c

d the fixed-end moments at c will be

$$M_{Fcb} = -\frac{(2.59)(30{,}000)(73{,}287)}{(144)(250)(250)}\Delta_1 = -635\Delta_1$$

$$M_{Fcd} = \frac{2.81}{2.59}(635\Delta_1) = 689\Delta_1$$

Similarly, for a horizontal displacement Δ_2 at point d only,

$$M_{Fcd} = -689\Delta_2$$

The correction of these fixed-end moments for rotation of the joints is given in Fig. 10.10*c*, together with the shears and reactive forces. By using the results recorded in Figs. 10.10*b* and *c* and the expressions for ΔH in the cables the following equilibrium equations can be written for the horizontal forces at joints *c* and *d*. For $\Sigma H_c = 0$,

$$-93.2 + 5.28\Delta_1 - 2.64\Delta_2 + 50.3\Delta_1 + 12 + 18.6\Delta_1 = 0$$

or

$$74.2\Delta_1 - 2.64\Delta_2 = 81.2$$

Also, for $\Sigma H_d = 0$

$$-42.0 - 2.64\Delta_1 + 1.32\Delta_2 + 18\Delta_2 + 5.2 + 9.4\Delta_2 = 0$$

or

$$-2.64\Delta_1 + 28.72\Delta_2 = 36.8$$

Solving these equations for Δ_1 and Δ_2 gives

$$\Delta_1 = 1.13 \text{ ft} \qquad \text{and} \qquad \Delta_2 = 1.38 \text{ ft}$$

From the results of this preliminary analysis, the deformation equations for the leeward cables should be revised. In fact, larger cables are recommended to reduce the stresses in the windward cables as well as the horizontal displacements Δ_1 and Δ_2. Additional refinements can be included such as the effect of the eccentricity of the cable reactions on the end moments in the shaft. The dynamic behavior of the tower due to the wind must, of course, be carefully studied and the references at the end of the chapter should be used for a further investigation of maximum and minimum loading conditions, wind pressures, and stability requirements.

10.14 Suspension Bridges

The analysis and design of suspension bridges frequently involve special problems that require more comprehensive treatment than is given in the following discussion. The fundamental equations for ordinary conditions will be considered here, and the reader who desires to continue his study of the subject further should consult the references listed at the end of the chapter. The suspension bridge as ordinarily constructed consists of the cables, towers, anchorages, stiffening trusses or girders, suspenders, and roadway. The arrangement shown in Fig. 10.11 has three spans where the stiffening trusses are hinged at the towers and where the cables are loaded in all spans. Other arrangements sometimes used include continuous stiffening trusses, unloaded cables in the end spans, self-anchored bridges, and various types of cables. For the conventional type of suspension bridge that is shown in Fig. 10.11, the

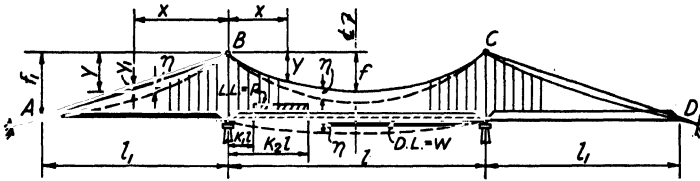



Fig. 10.11

following assumptions are commonly made in the analysis:

1. The dead weight is uniformly distributed and is carried entirely by the cable. For this loading condition the shape of the cable is almost parabolic and the cable must be placed in an assigned position during construction.
2. The deflection η of the stiffening truss for live loads is the same as for the cable. Any change in length of the suspenders is therefore neglected.
3. The proportion of the live load that is taken by the cable, designated in Fig. 10.12 by q, is often assumed to be uniformly distributed in order to simplify the mathematical solution. This assumption means that the shape of the cable is taken as a parabola for combined dead and live loads. The error involved in this assumption depends on the ratio of live to dead load as well as the physical characteristics of the cable and stiffening truss. For large structures where the dead weight is a major portion of the total, the error is small.

   In the solution in which the displacements are expressed by trigonometric series no assumption as to the distribution of q is necessary. For this reason, as well as others, such a solution has many advantages.
4. The horizontal component H of the cable stress is assumed to be constant for all spans unless the cable is fixed to unusually stiff towers.
5. In calculating the external work that occurs when the live load is applied to the structure, it is assumed that the deflection is proportional to the applied live load. This assumption is permissible for the ratio of live to dead load and for the sag-span ratios ordinarily used.

10.15 Calculation of the Horizontal Component H

The horizontal component H of the stress in the cable is usually determined first in the analysis of suspension bridges. From assumption 1 in Article 10.14 the uniform dead load w is taken entirely by the cable and

the value of H_D can therefore be calculated directly from equilibrium conditions. Taking moments about point B (Fig. 10.12) we obtain

$$H_D = \frac{wl^2}{8f} \tag{10.43a}$$

where f = maximum cable sag in center span
l = length of center span
w = uniformly distributed dead load

The equation of the cable for point B as origin is

$$y = \frac{4fx}{l^2}(l - x) = \frac{wx}{2H_D}(l - x) \tag{10.43b}$$

The sag-span ratio f/l is known from preliminary studies. In the spinning of the cables the wires must be set at the proper elevation so that the desired sag-span ratio is obtained at normal temperatures after the roadway is completed. Considerable calculations are required to determine the correct elevation of the wires during construction (see Ref. 2).

Since the live load is carried by both the stiffening truss and the cable, the structure is once statically indeterminate for this loading condition. The additional horizontal component H_L in the cable, because of the live load and temperature changes, is usually selected as the redundant quantity to be determined. The total horizontal component H is therefore equal to

$$H = H_D + H_L = H_D(1 + \beta) \tag{10.44}$$

where $\beta = H_L/H_D$.

It will be shown that the numerical value of H depends on the change in temperature as well as on the magnitude and distribution of the live

Fig. 10.12

and dead loads. By equating the external work done on the cable to the corresponding strain energy existing in the cable, an equation is obtained from which the value of the horizontal component H can be determined by trial. The external work, for any portion ds of the cable, that is caused by applying an additional increment of load q to the dead load w is

$$dW_e = w\eta\, dx + \frac{q}{2}\eta\, dx = \left(w + \frac{q}{2}\right)\eta\, dx \tag{10.45a}$$

This equation obviously implies a linear relation between q and the displacement η. Such a linear variation does not theoretically exist, and the only justification for using it is that the displacements that are calculated from a more exact analysis are found to be practically linear for actual structures (see Ref. 8). This assumption would not be warranted for the analysis of the horizontal wire in Article 10.2, for in such a member the displacements are far from being proportional to the load.

The total external work for all spans is

$$W_e = \sum \int_0^l \left(w + \frac{q}{2}\right)\eta\, dx \tag{10.45b}$$

in which the Σ means that the integration is made for all loaded spans.

The internal work or strain energy in any element ds of the cable is

$$dW_i = \left(T_D + \frac{T_L}{2}\right)\left(\frac{T_L}{AE} + \alpha t\right) ds \tag{10.46a}$$

in which the tension T in the cable at any section is equal to

$$T = H \sec\theta = H\frac{ds}{dx} \tag{10.46b}$$

αt = deformation per unit length due to change in temperature. Therefore

$$dW_i = \left(H_D + \frac{H_L}{2}\right)\frac{ds}{dx}\left(\frac{H_L}{AE}\frac{ds}{dx} + \alpha t\right) ds$$

or

$$dW_i = H_D\left(1 + \frac{\beta}{2}\right)\left[\frac{H_D}{AE}\beta\left(\frac{ds}{dx}\right)^2 + \alpha t\frac{ds}{dx}\right] ds \tag{10.46c}$$

The total internal work for all spans is

$$W_i = H_D\left(1 + \frac{\beta}{2}\right)\sum\int_0^l \left[\frac{H_D}{AE}\beta\left(\frac{ds}{dx}\right)^2 + \alpha t\frac{ds}{dx}\right] ds \tag{10.47}$$

in which the integration is made for all spans.

For equilibrium of any portion of the cable (see Article 10.3),

$$H_D \frac{d^2y}{dx^2} = -w \tag{10.48a}$$

and

$$(H_D + H_L)\frac{d^2(y + \eta)}{dx^2} = -(w + q) \tag{10.48b}$$

from which

$$H_D \frac{d^2y}{dx^2} + H_D \frac{d^2\eta}{dx^2} + H_L \frac{d^2y}{dx^2} + H_L \frac{d^2\eta}{dx^2} = -w - q \tag{10.48c}$$

Substituting the value of w in equation 10.48*a* into equation 10.48*c* gives

$$q = -\frac{H_L}{H_D} H_D \frac{d^2y}{dx^2} - H_D(1 + \beta)\frac{d^2\eta}{dx^2}$$

or

$$q = \beta w - H_D(1 + \beta)\frac{d^2\eta}{dx^2} \tag{10.48d}$$

By equating W_i to W_e and eliminating q by equation 10.48*d*

$$H_D\left(1 + \frac{\beta}{2}\right)\sum\int_0^l \left[\frac{H_D}{AE}\beta\left(\frac{ds}{dx}\right)^2 + \alpha t\frac{ds}{dx}\right] ds$$
$$= \sum\int_0^l \left[w + \frac{w\beta}{2} - \frac{H_D(1 + \beta)}{2}\frac{d^2\eta}{dx^2}\right]\eta\, dx \tag{10.49}$$

from which the value of β can be calculated by trial after the displacement η has also been expressed in terms of β.

In solving equation 10.49 it is necessary to evaluate the integrals

$$\int\left(\frac{ds}{dx}\right)^2 ds \qquad \text{and} \qquad \int\left(\frac{ds}{dx}\right) ds$$

which can be written in the form

$$\int\left(\frac{ds}{dx}\right)^3 dx \qquad \text{and} \qquad \int\left(\frac{ds}{dx}\right)^2 dx$$

for all spans.

For a symmetrical center span the equation for the cable will be simplified if an abscissa x' is measured from the center of the span and y' is measured from the horizontal through the vertex, that is, y is equal to $f - y'$ and x is equal to $l/2 + x'$. For this change in the origin of the coordinates, equation 10.43*b* becomes

$$f - y' = \frac{4f}{l^2}\left(\frac{l^2}{4} - x'^2\right) = f - \frac{4fx'^2}{l^2}$$

and therefore,

$$y' = \frac{4f}{l^2}x'^2$$

$$\frac{dy'}{dx'} = \frac{8f}{l^2}x'$$

$$ds = \sqrt{dx'^2 + dy'^2} = dx'\left[1 + \left(\frac{dy'}{dx'}\right)^2\right]^{1/2}$$

$$\left(\frac{ds}{dx'}\right)^3 dx' = \left(1 + \frac{64f^2}{l^4}x'^2\right)^{3/2} = (1 + a)^{3/2}$$

where, if f/l is replaced by m

$$a = \frac{64f^2}{l^4}x'^2 = \frac{64m^2}{l^2}x'^2$$

Expanding $(1 + a)^{3/2}$ by the binomial theorem gives, for $a \leq 1$

$$(1 + a)^{3/2} = 1 + \tfrac{3}{2}a + \tfrac{3}{8}a^2 - \tfrac{1}{16}a^3 + \tfrac{3}{128}a^4 \ldots$$

or, if $m = f/l$

$$\left(1 + \frac{64m^2}{l^2}x'^2\right)^{3/2} = 1 + \frac{96m^2}{l^2}x'^2 + \frac{(64)(24)m^4}{l^4}x'^4 - \frac{(4)(64)(64)m^6}{l^6}x'^6 + \ldots$$

If the terms of the series are integrated with respect to x', we obtain

$$\int_{-l/2}^{+l/2}\left(\frac{ds}{dx'}\right)^3 dx' = x' + \frac{32m^3}{l^2}x'^3 + \left.\frac{(64)(24)m^4}{l^4}\frac{x'^5}{5} - \frac{(4)(64)(64)m^6}{l^6}\frac{x'^7}{7}\right]_{-l/2}^{+l/2}$$

$$= l[1 + 8m^2 + 19.2m^4 - 36.6m^6 + \ldots] \qquad (10.50a)$$

which gives sufficiently accurate results for all practical values of m.

Similarly, the integral

$$\int_{-l/2}^{+l/2}\left(\frac{ds}{dx'}\right)^2 dx' = \int_{-l/2}^{+l/2}\left(1 + \frac{64m^2}{l^2}x'^2\right) dx'$$

$$= \left(x' + \frac{64m^2}{2}\frac{x'^3}{3}\right)_{-l/2}^{+l/2}$$

$$= l[1 + \tfrac{16}{3}m^2] \qquad (10.50b)$$

For loaded side spans (see Fig. 10.14)

$$\int_0^l \left(\frac{ds}{dx'}\right)^3 dx' = \int_{x_a'}^{x_b'} \left(1 + \frac{64f'^2}{l_1'^4}\, x'^2\right)^{3/2} dx' \qquad (10.51a)$$

$$\int_0^l \left(\frac{ds}{dx}\right)^2 dx' = \int_{x_a'}^{x_b'} \left(1 + \frac{64f'^2}{l_1'^4}\, x'^2\right) dx' \qquad (10.51b)$$

These integrals can be evaluated by the binomial theorem as was done above for the center span.

10.16 Value of Displacement η for Center Span

Before equation 10.49 can be solved for β, the terms η and $d^2\eta/dx^2$ must be expressed in terms of β, the dead load w, the live load p, and the properties of the stiffening truss. This relation is obtained from the fundamental equation of the elastic curve of the stiffening truss in practically the same manner as for the beams in Article 10.8. From Fig. 10.12, the bending moment due to the live load p at any section through the stiffening truss at a distance x from the left end is

$$M = M_s - (H_D + H_L)\eta - H_L y \qquad (10.52a)$$

This equation can be checked by taking moment of the forces shown in Fig. 10.13 about point O' on the displaced position of the cable, which gives

$$V_D'x + (V_L' + V_L'')x - \frac{wx^2}{2} - \frac{pz^2}{2} - (H_D + H_L)(y + \eta) - M = 0$$

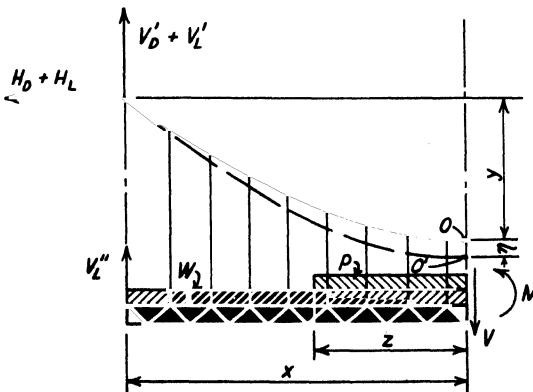



Fig. 10.13

However,

$$V_D'x - \frac{wx^2}{2} - H_D y = 0$$

in accordance with the assumptions in Article 10.14 for the dead load condition. For a symmetrical center span, the terms $(V_L' + V_L'')x - pz^2/2$ are equal to the bending moment M_s at any section at a distance x from the left end in a simply supported span L with the corresponding load p lb per linear foot. Therefore the moment M in the stiffening truss is expressed by equation 10.52*a*.

Considering the stiffening truss as a beam

$$\frac{d^2\eta}{dx^2} = -\frac{M}{EI} = \frac{-M_s + (H_D + H_L)\eta + H_L y}{EI} \tag{10.52b}$$

If we let

$$j^2 = \frac{H_D + H_L}{EI} = \frac{H_D(1 + \beta)}{EI} \tag{10.52c}$$

then

$$\frac{d^2\eta}{dx^2} - j^2\eta = -\frac{M_s}{EI} + \frac{H_L y}{EI} \tag{10.52d}$$

The solution of this equation, as given in Article 10.8, is

$$\eta = A \cosh jx + B \sinh jx + \frac{1}{EI}\left(\frac{M_s}{j^2} + \frac{1}{j^4}\frac{d^2M_s}{dx^2} + \ldots\right) - \frac{H_L}{EI}\left(\frac{y}{j^2} + \frac{1}{j^4}\frac{d^2y}{dx^2} + \ldots\right) \tag{10.53}$$

The integration constants A and B must satisfy the boundary conditions and the conditions of continuity of the elastic curve for any particular loading. For a partial uniform loading on the roadway, the bending moment M_s and therefore the displacement η must be expressed by three different equations, and consequently, there are six integration constants to evaluate. This requirement makes the exact solution of the differential equation extremely laborious. An alternative solution, as has already been seen, is to express the displacement η by a trigonometric series, as was done by S. Timoshenko and other investigators. (See Ref. 5.)

As in Article 10.10, the displacement η can be expressed by the series

$$\eta = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{2\pi x}{l} + \ldots = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \tag{10.54a}$$

$$\frac{d^2\eta}{dx^2} = -\frac{\pi^2}{l^2}\sum_{n=1}^{\infty} n^2 a_n \sin\frac{n\pi x}{l} \tag{10.54b}$$

In Article 10.10 it was shown that the bending moment in a simply supported beam for a concentrated load W at a distance c from the left support is expressed by the Fourier series

$$M_s = \frac{2Wl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi c}{l}\sin\frac{n\pi x}{l} \tag{10.55}$$

A uniform load that extends from k_1l to k_2l (Fig. 10.12) is equivalent to a number of concentrated loads $p\,dz$ which are at a distance $z = c$ from the left support. Therefore the moment caused by the uniform load is

$$M_s = \frac{2pl}{\pi^2}\int_{z=k_1l}^{z=k_2l}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi z}{l}\,dz\sin\frac{n\pi x}{l} \tag{10.56a}$$

which, after integration, gives

$$M_s = \frac{2pl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(\cos n\pi k_1 - \cos n\pi k_2)}{n^3}\sin\frac{n\pi x}{l} \tag{10.56b}$$

The equation for the original position of the cable

$$y = \frac{4f}{l^2}(lx - x^2) = \frac{w}{2H_D}(lx - x^2) \tag{10.57a}$$

is readily developed into the Fourier series

$$y = \frac{2wl^2}{H_D\pi^3}\sum_{n=1}^{\infty}\frac{(1 - \cos n\pi)}{n^3}\sin\frac{n\pi x}{l} \tag{10.57b}$$

When Fourier series for η, $d^2\eta/dx^2$, M_s, and y are substituted in equation 10.52d, we obtain

$$\begin{aligned} -\frac{\pi^2}{l^2}\sum_{n=1}^{\infty}n^2a_n\sin\frac{n\pi x}{l} - \frac{H_D(1+\beta)}{EI}\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{l} \\ = -\frac{2pl^2}{\pi^3EI}\sum_{n=1}^{\infty}\frac{(\cos n\pi k_1 - \cos n\pi k_2)}{n^3}\sin\frac{n\pi x}{l} \\ + \frac{2H_Lwl^2}{EIH_D\pi^3}\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n^3}\sin\frac{n\pi x}{l} \end{aligned} \tag{10.58a}$$

For any value of $\sin(n\pi x/l)$ that is, for any value of the integer n, equation 10.58a requires that the coefficients a_n have the value

$$\begin{aligned} -a_n\left[\frac{n^2\pi^2}{l^2} + \frac{H_D(1+\beta)}{EI}\right] = -\frac{2pl^2}{\pi^3EI}\left(\frac{\cos n\pi k_1 - \cos n\pi k_2}{n^3}\right) \\ + \frac{2H_Lwl^2}{EIH_D\pi^3}\left(\frac{1-\cos n\pi}{n^3}\right) \end{aligned} \tag{10.58b}$$

or since $H_L/H_D = \beta$

$$a_n = \frac{2l^4[p(\cos n\pi k_1 - \cos n\pi k_2) - \beta w(1 - \cos n\pi)]}{n^3\pi^3[n^2\pi^2 EI + H_D(1 + \beta)l^2]} \tag{10.58c}$$

This expression for the coefficients a_n contains the term β which is also the unknown quantity in equation 10.49. However, as equation 10.49 must be solved by trial the calculation of the coefficients a_n for any assumed value of β does not add much work to the entire solution. Once equation 10.49 has been solved for β, the cable stress, deflections, bending moments, and shear in the stiffening truss are readily ascertained.

In solving equation 10.49, it is necessary to transform the term

$$\int_0^l \frac{H_D(1 + \beta)}{2} \eta \frac{d^2\eta}{dx^2} dx$$

by means of equations 10.54*a*, *b* to the form

$$\frac{H_D(1 + \beta)}{2} \int_0^l - \frac{n^2\pi^2}{2} a_n^2 \sin^2 \frac{n\pi x}{l} dx$$

which, for any particular value of n, is equal to

$$\frac{H_D(1 + \beta)}{2} \left[-\left(\frac{n^2\pi^2}{l^2} a_n^2 \right) \left(\frac{l}{2} \right) \right] \tag{10.59a}$$

Consequently, the value of the term for all values of a_n is

$$- \frac{H_D(1 + \beta)\pi^2}{4l} \sum_{n=1}^{\infty} n^2 a_n^2 \tag{10.59b}$$

The reader should be able to make the necessary evaluation of the integral

$$w\left(1 + \frac{\beta}{2}\right) \int_0^l \eta \, dx$$

10.17 *Value of the Displacement η for the Side Spans*

For the side spans the equilibrium of the cable requires that

$$H_D \frac{d^2y}{dx^2} = -w_1 \tag{10.60a}$$

which after integrating twice gives

$$H_D y = - \frac{w_1 x^2}{2} + C_1 x + C_2 \tag{10.60b}$$

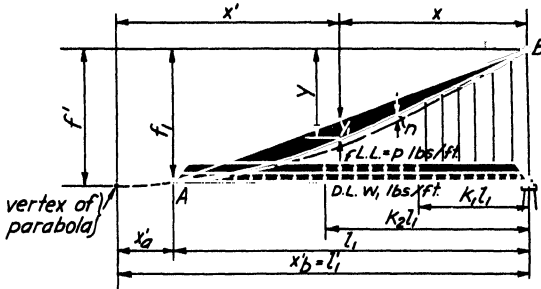



Fig. 10.14

If point B (Fig. 10.14) at the top of the tower is selected as the origin,

$$y = 0 \qquad \text{when } x = 0 \qquad \text{or} \qquad C_2 = 0$$

$$y = f_1 \qquad \text{when } x = l_1 \qquad \text{or} \qquad C_1 = H_D\frac{f_1}{l_1} + \frac{w_1 l_1}{2}$$

Therefore

$$H_D\left(y - \frac{f_1}{l_1}x\right) = -\frac{w_1 x^2}{2} + \frac{w_1 l_1}{2}x \tag{10.60c}$$

but

$$y - \frac{f_1}{l_1}x = y_1$$

Consequently, the equation for the shape of the cable is

$$y_1 = \frac{w_1}{2H_D}(l_1 x - x^2) \tag{10.60d}$$

This equation is identical in form with equation 10.57a, and therefore if y is replaced by y_1, l by l_1, I by I_1, the equation for the elastic curve of the stiffening truss in a side span can be obtained in the same manner as for the center span. That is, if for the side span

$$\eta = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l_1} \tag{10.61a}$$

then

$$b_n = \frac{2l_1^4[p(\cos n\pi k_1 - \cos n\pi k_2) - \beta w_1(1 - \cos n\pi)]}{n^3\pi^3[n^2\pi^2 EI_1 + H_D(1 + \beta)l_1^2]} \tag{10.61b}$$

In addition, as for the center span, the term

$$\int_0^{l_1} \frac{H_D(1 + \beta)}{2}\eta\frac{d^2\eta}{dx^2}dx = -\frac{H_D(1 + \beta)\pi^2}{4l_1}\sum_{n=1}^{\infty} n^2 b_n \tag{10.61c}$$

Example 10.10 The internal forces in one of the cables and stiffening trusses of the Detroit-Windsor bridge will be obtained by the equations

℄ of span
925.0'
$\alpha_1 = 20°32'$
$\alpha_2 = 24°-0'$
f = 205.6'
p
w
.25 ℓ
.75 ℓ
$\ell_1 = 984.2'$
$\ell = 1850.0'$
$\ell_2 = 833.9'$
Value of β'
Value of β
t = -50°F
t = 0
β = β'
β = .206
β = .219
.60
.50
.40
.30
.20
.10
0
.16
.17
.18
.19
.20
.21
.22

(a)

(b)

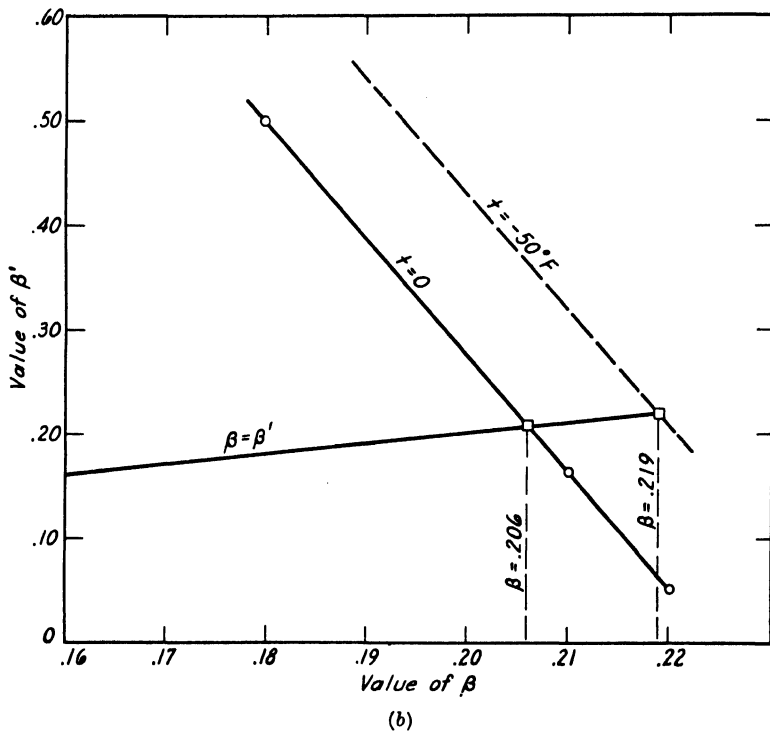
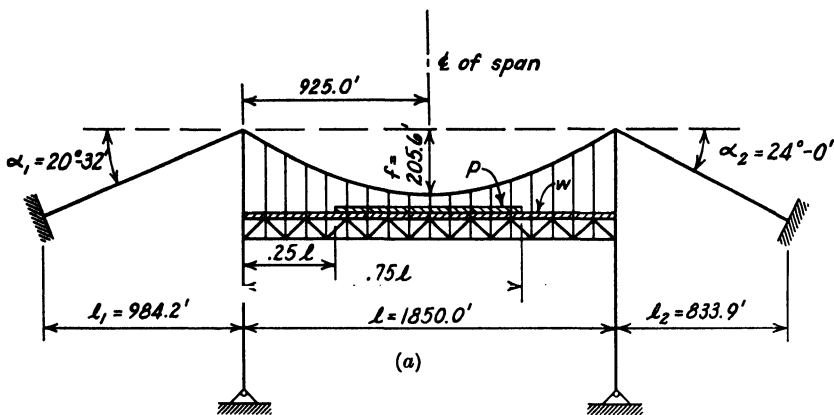



Fig. 10.15

in Articles 10.15 and 10.16. The arrangement and properties of the east cable as given by A. A. Jakkula in his paper "The Theory of the Suspension Bridge," *Publ. Int. Assoc. for Bridge and Struct. Eng.*, 1936, are shown in Fig. 10.15*a*. In addition to the dimensions given in the diagram, the following data will be used.

$A_c = 240.9$ in.^{2} $\qquad E_c = 27 \times 10^3$ kips per square inch

$E_t = 30 \times 10^3$ kips per square inch $\qquad I_t = 113.7$ ft^4

$w = 6.2$ kips per foot $\qquad p = 2.0$ kips per foot $\qquad k_1 = 0.25 \qquad k_2 = 0.75$

Since this bridge has unloaded backstays, only the deflection η of the center span need be considered and the cable in the end spans is assumed to be straight. For this particular problem equation 10.49 will be written in the form

$$\beta' = \frac{w\left(1 + \frac{\beta}{2}\right)\int_0^l \eta\,dx - \frac{H_D(1+\beta)}{2}\int_0^l \eta\frac{d^2\eta}{dx^2}dx - H_D\left(1+\frac{\beta}{2}\right)\alpha t\sum\int\left(\frac{ds}{dx}\right)^2 dx}{\frac{H_D^2}{A_cE_c}\left(1+\frac{\beta}{2}\right)\sum\int\left(\frac{ds}{dx}\right)^3 dx}$$

This equation can be further simplified since $\int_0^l \left(\frac{ds}{dx}\right)^3 dx$ for the center span ($m = \frac{1}{9}$) as given by equation 10.50*a* is

$$1850\left[1 + \frac{8}{81} + \frac{19.2}{(81)(81)} - \frac{36.6}{(81)^3}\right] = 2038$$

For the straight end cables this integral is equal to $l_1 \sec^3 \alpha_1$ and $l_2 \sec^3 \alpha_2$, respectively. Therefore the sum of the three integrals is

$$\sum\int\left(\frac{ds}{dx}\right)^3 dx = 2038 + 1198 + 1094 = 4330 \text{ ft}$$

The sum of the integrals involving $(ds/dx)^2\,dx$ can be evaluated as follows.

Center span (equation 10.50*b*) $1850(1 + \frac{16}{243}) = 1972$

End span l_1 $\qquad l_1 \sec^2 \alpha_1 = 1122$

End span l_2 $\qquad l_2 \sec^2 \alpha_2 = 1000$

or

$$\sum\int\left(\frac{ds}{dx}\right)^2 dx = 4094$$

The integrals involving η and the $d^2\eta/dx^2$ can be expressed in terms of the series coefficients a_n thus:

$$\int_0^l \eta\, dx = a_n \int_0^l \sin\frac{n\pi x}{l}\, dx = -\frac{a_n l}{n\pi}\cos\frac{n\pi x}{l}\Bigg|_0^l$$

$$= \frac{2a_n l}{n\pi} \qquad \text{for } n = 1, 3, 5, 7, \ldots$$

and zero for even values of n.

Therefore

$$w\left(1 + \frac{\beta}{2}\right)\int_0^l \eta\, dx = \frac{2\omega l}{\pi}\left(1 + \frac{\beta}{2}\right)\sum_{n=1,3,5,\ldots} \frac{a_n}{n}$$

From equation 10.54*c*,

$$\frac{H_D(1+\beta)}{2}\int_0^l \eta\frac{d^2\eta}{dx^2}\, dx = -\frac{H_D(1+\beta)}{4l}\pi^2 \sum_{n=1,3,5,\ldots}^{\infty} n^2 a_n^{\,2}$$

For this problem k_1 and k_2 will be taken as $\frac{1}{4}$ and $\frac{3}{4}$, respectively and equation 10.58*c* can be reduced to the form

$$a_n = \frac{2[\cos(n\pi/4) - \cos(3n\pi/4)] - 6.2\beta(1 - \cos n\pi)}{0.00642n^5 + 0.05843(1+\beta)n^3}$$

From this expression values of a_n can be calculated for any value of β and any integral value of n. It can be seen that the numerator is zero for all even values of n, which will be the case for any symmetrical loading condition.

The arrangement shown in Table 10.4 is convenient for ordinary calculations and give the values of a_n for β values of 0.18 and 0.22. If all the 2 kips per foot is assumed to be carried by the cable,

$$H_L = \frac{3pl^2}{32f} = \frac{(3)(2.0)(1850)^2}{(32)(205.6)} = 3125 \text{ kips}$$

$$\beta = \frac{H_L}{H_D} = \frac{3125}{12{,}900} = 0.242$$

Therefore the actual value of β must be less than 0.24 and it will be assumed to lie somewhere between 0.18 and 0.22.

Table 10.4

n	a_n	$\beta = 0.18$	$\beta = 0.22$
1	$a_1 = \dfrac{2.828 - 12.4\beta}{0.06485 + 0.05845\beta} =$	7.9077	1.2868
3	$a_3 = \dfrac{-2.828 - 12.4\beta}{3.13767 + 1.5776\beta} =$	−1.4788	−1.5944
5	$a_5 = \dfrac{-2.828 - 12.4\beta}{27.3663 + 7.3038\beta} =$	−0.1764	−0.1918
7	$a_7 = \dfrac{2.828 - 12.4\beta}{127.9424 + 20.0415\beta} =$	0.0045	0.0008

Table 10.5

n	a_n	$\frac{a_n}{n}$	$n^2 a_n$	$n^2 a_n^2$
		$\beta = 0.18$		
1	7.9077	7.9077	7.9077	62.5317
3	−1.4788	−0.4929	−13.3092	19.6816
5	−0.1764	−0.0353	−4.4100	0.7779
7	0.0045	0.0006	0.2205	0.0010
		7.3801	−9.591	82.9922
		$\beta = 0.22$		
1	1.2868	1.2868	1.2868	1.6559
3	−1.5944	−0.5315	−14.3496	22.8790
5	−0.1918	−0.0384	−4.7950	0.9197
7	0.0008	0.0001	0.0392	0.0000
		0.7170	−17.8186	25.4546

If β is assumed as 0.18,

$$\frac{2wl}{\pi}\left(1 + \frac{\beta}{2}\right)\sum \frac{a_n}{n} = \frac{(2)(6.2)(1850)}{\pi}(1.09)(7.3801) = 58{,}740$$

$$\frac{H_D(1+\beta)}{2}\int_0^l \eta \frac{d^2\eta}{dx^2}\,dx = -\frac{H_D(1+\beta)\pi^2}{4l}\sum n^2 a_n^2$$

$$= -\frac{(12{,}900)(9.87)(1.18)}{(4)(1850)}(82.9922) = -1684.6$$

$$\frac{H_D^2}{A_c E_c}\left(1 + \frac{\beta}{2}\right)\sum \int \left(\frac{ds}{dx}\right)^3 dx = \frac{(12{,}900)^2(1.09)}{(240.9)(27)(10^3)}(4330) = 120{,}754$$

Solving for β' gives

$$\beta' = \frac{58{,}740 + 1684.6}{120{,}754} = 0.5004$$

Repeating the preceding calculations for an assumed β of 0.22 gives

$$\beta' = \frac{5811 + 534}{122{,}968} = 0.0516$$

The true value of β may be obtained from Fig. 10.15*b* by assuming that the relations between β and β' is linear as shown by A. A. Jakkula in Ref. 8. For practical bridges this linear relationship is almost exact. To determine the point where β' is equal to β, construct the line $\beta' = \beta$ as shown in Fig. 10.15*b*. The intersection of the two lines gives the true value of β as 0.206.

The bending moments in the stiffening truss are given by equation 10.52*a* in the form

$$M = M_s - (H_D + H_L)\eta - H_L y$$

and by differentiating this expression with respect to x the expression for the shear becomes

$$V = \frac{dM}{dx} = V_s - (H_D + H_L)\frac{d\eta}{dx} - H_L\frac{dy}{dx}$$

Using a β value of 0.206 as previously determined, the preceding equations give

$$M = M_s - 15{,}557\eta - 2657y$$

$$V = V_s - 15{,}557\frac{d\eta}{dx} - 2657\frac{dy}{dx}$$

in which M_s and V_s are the bending moment and shear for the live load on a simply supported truss and

$$\eta = \sum a_n \sin\frac{n\pi x}{l}, \qquad y = \frac{4f}{l^2}x(l - x)$$

$$\frac{d\eta}{dx} = \frac{\pi}{l}\sum na_n \cos\frac{n\pi x}{l}, \qquad \frac{dy}{dx} = \frac{4f}{l^2}(l - 2x)$$

The numerical data necessary for determining the shears and bending moments in the stiffening truss are summarized in Tables 10.6, 10.7, and 10.8.

The bending-moment and shear-diagrams are shown in Fig. 10.16 together with q, the variation in hangar force, as given by equation 10.48*d*.

Table 10.6

(n)	a_n ($\beta = 0.206$)	na_n	n^2a_n	$n^2a_n^2$
1	3.5635	3.5635	3.5635	12.6985
3	−1.5543	−4.6629	−13.9887	21.7426
5	−0.1864	−0.9320	−4.6600	0.8686
7	0.0021	0.0147	0.1029	0.0002
			$\sum =$ −14.9823	35.3099

Table 10.7

$\frac{x}{l}$	η	y	$\sum na_n \cos \frac{n\pi x}{l}$	$2657 \frac{dy}{dx} = 1181.1\left(1 - \frac{2x}{l}\right)$
0	0	0	−2.0167	1181.1
$\frac{1}{8}$	−0.3107	89.95	1.5299	885.8
$\frac{1}{4}$	1.5508	154.2	6.4854	590.6
$\frac{3}{8}$	3.9605	192.75	4.8051	295.3
$\frac{1}{2}$	4.9293	205.6	0	0

Table 10.8

x	M_s (ft-kips)	V_s (kips)	M (ft-kips)	V (kips)
0	0	925	0	−202.8
$\frac{1}{8}$	213,906	925	−20,257	−1.2
$\frac{1}{4}$	427,812	925	−6,023	163.1
$\frac{3}{8}$	588,241	462.5	14,491	40.3
$\frac{1}{2}$	641,718	0	18,754	0

10.18 Flexible Arches

In Chapter 9 the analysis of arches and curved members was made by the elastic theory; that is, the displacement of the arch axis was neglected. However, in the design of long-span arch bridges the stresses calculated by the elastic theory are found to be considerably less than their true values as given by the deflection theory, which considers the movement of the arch axis. An exact mathematical solution of an arch rib by the deflection theory is similar to the analysis of a suspension bridge, although

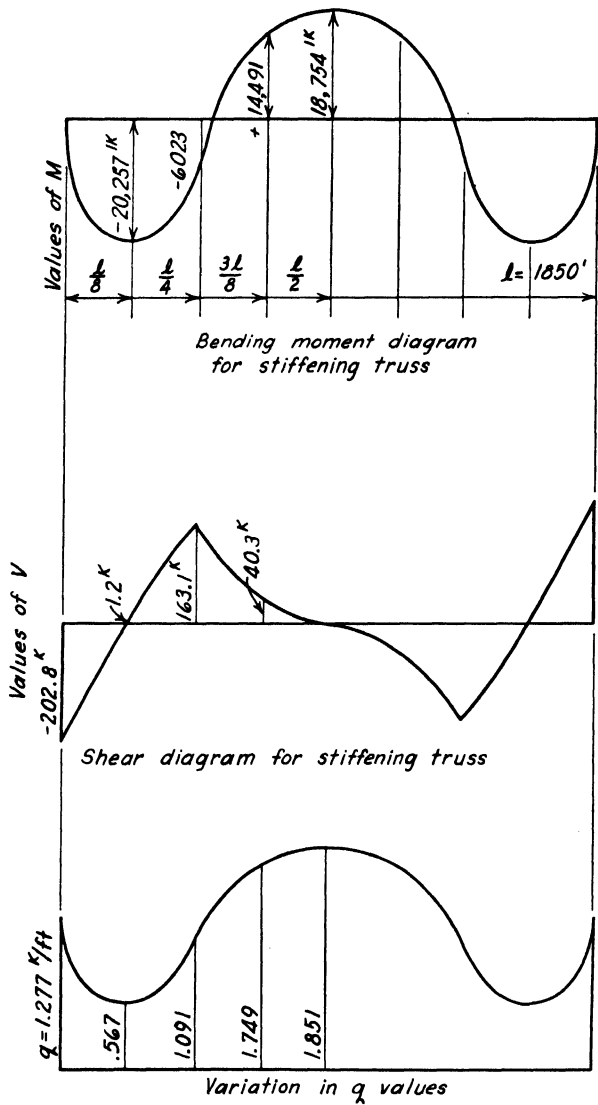



Fig. 10.16

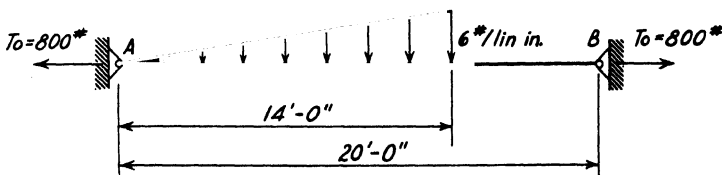
the numerical calculations are even more laborious and difficult. The increased difficulty arises in the expression for the internal work in the arch rib, which involves both flexural and direct stresses, whereas the internal work in the cable of a suspension bridge is due only to axial tension. To mitigate these mathematical difficulties many investigators have confined their analysis to special problems in which the equations

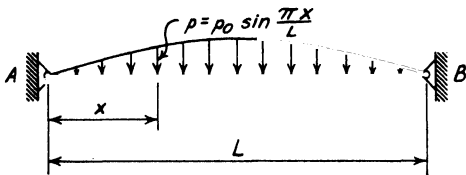
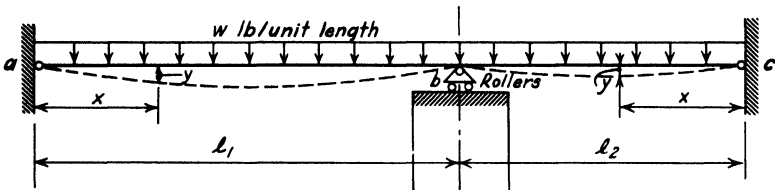
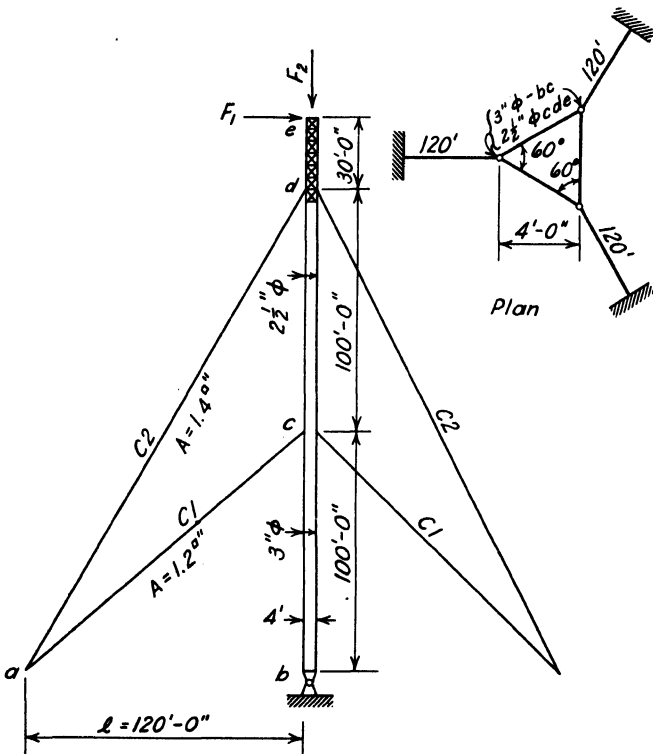
can be simplified. Despite such assumptions the numerical work involves tremendous effort. The references at the end of the chapter will be found useful for further study of these problems.

Two of the longest solid rib arch bridges that have yet been built are the Rainbow Arch bridge (Ref. 18) and the Lewiston-Queenston bridge over the Niagara River which have spans of 950 and 1000 ft, respectively. It is interesting to note that the maximum positive bending moment at the quarter point of a two-hinged arch rib that was first investigated for the Rainbow bridge was 63% more for the deflection theory than for the elastic theory. For the fixed arch rib that was actually constructed this moment was about 18% larger by the deflection theory. It is apparent that for structures of this magnitude the deflection of the arch rib must be considered.

In the analysis of the Rainbow Arch bridge, a method of successive approximations was found to be most advantageous. In most solutions by this method, the values of the reactions, shears, and bending moments as determined by the elastic theory are used as the first approximation, and from them a trial position of the arch axis is determined by either algebraic or graphical methods. The coordinates of the various elements into which the arch axis is divided are then corrected and the numerical operations repeated. The corrections become less for each repetition until the difference can be neglected. This work can usually be reduced by anticipating some of the corrections in advance and by programming the solution for computer analysis. The practical estimation of such corrections is described by the designers of the Rainbow Arch bridge in Ref. 18. Every design of bridges of this type will require special treatment, particularly with respect to the problems of fabrication and erection.

Problems

10.1 If a horizontal wire of 20′-0″ span and a cross-sectional area A of 0.06 in.^{2} has an initial tension of 800 lb and is subjected to a linearly varying radial pressure as shown, determine the maximum tension and vertical displacement. Use $E = 27 \times 10^6$ psi and neglect the weight of the wire.

Problem 10.1

10.2 Solve the problem in Example 10.1 if the uniform load is applied vertically instead of radially. Assume p is weight per horizontal distance.

Problem 10.3

Problem 10.4

Problems 10.5 and 10.10

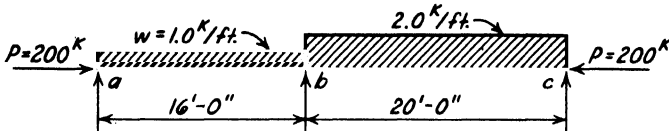
10.3 Determine the maximum tensile force in a wire with area A and modulus E if the radial pressure is equal to $p = p_0 \sin \pi x/L$.

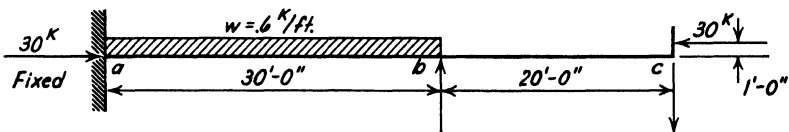
10.4 The wires ab and bc are supported so that points a and c do not move, but point b can move horizontally. The uniform vertical pressure w, area A, and modulus E are constant for both spans. The horizontal spans are l_1 and l_2 respectively. Assuming that the sag is sufficiently small so that the tension T can be approximated by the horizontal component H, express H and the maximum sag in each span in terms of w, l_1, l_2, and AE.

10.5 The catenary cables $C1$ in the guyed tower shown have an initial tension such that H_0 is 20,000 lb and are subjected to a uniform load q along the cable of 6 lb per foot. If the area of the cable is 1.2 in.^{2} and E is 25×10^6 psi, determine the tension T at both ends, the vertical component at the upper end c, the maximum sag s, and the total length L.

10.6 Determine the change in the horizontal component H of the cable $C1$ in Problem 10.5 for a drop in temperature of 60°F in both cable and tower. Use $\alpha = 6.7 \times 10^{-6}$ per °F. Assume that the horizontal distance l does not change.

10.7 Calculate the change in the horizontal component H for changes in the horizontal projection Δl of the cable in Problem 10.5 of $\Delta l = \pm 0.4$ ft.

10.8 Construct the bending-moment diagram for member bc of the continuous beam shown if the value of $EI = 60 \times 10^6$ lb-ft^2. Give value of maximum and minimum moments.

Problem 10.8

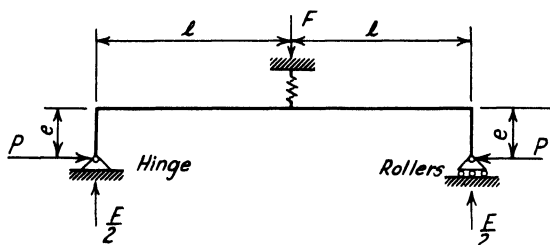
10.9 The member abc is fixed at a, continuous over b, and held against only vertical displacement at c. Calculate all end moments and the maximum and minimum moments in both spans. Use $EI = 10^4$ kips-ft^2 for both spans.

Problem 10.9

10.10 The 230-ft guyed tower shown in Problem 10.5 has a triangular cross section and is braced by six cables. Determine the axial loads and bending moments in the tower for a uniform wind load of 200 lb per linear foot and concentrated loads at the top of $F_1 = 4000$ lb and $F_2 = 5000$ lb. Use properties for cables $C1$ as given in Problem 10.5, and for $C2$ use $A = 1.4$ in.^{2}, $q = 6.5$ lb per foot, and $H_0 = 15,000$ lb. Neglect the pressure of the wind on the cables.

10.11 If a compression member of length $2l$ (see figure) is laterally supported at the center by a member with a spring constant K lb per inch, show by Equation 10.25 that the force F in the spring is equal to

$$F = \frac{2Pea(1 - \cos al)}{\sin al - (l - 2P/K)a \cos al}$$

in which

$$a = \sqrt{P/EI}$$

Problem 10.11

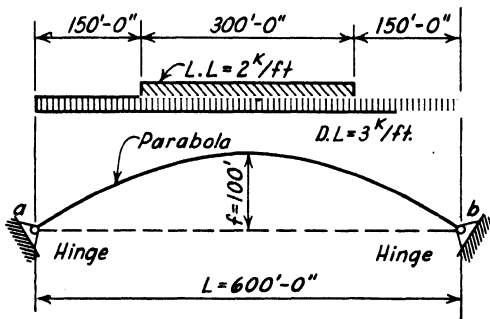
10.12 Solve the problem in Example 10.10 if $k_1 = 0$ and $k_2 = 0.5$, that is, the live load is applied to the left half of the main span. Assume no change in temperature.

10.13 Solve for β in Example 10.10 for the same loading arrangement and with a temperature change of $-50°$F. Use a coefficient of linear expansion of 6.5×10^{-6}.

10.14 A two-hinged arch with a parabolic axis and of constant cross section has the following proportions:

$$L = 600'\text{-}0'', \quad f = 100'\text{-}0'', \quad A = 400 \text{ in.}^2$$
$$I = 80 \times 10^4 \text{ in.}^4 \quad E = 29 \times 10^6 \text{ psi}$$

Using the dead and live loads shown in the diagram, compare the bending moment at the crown of the arch for the elastic theory with the value obtained by the deformation theory.

(*Suggestion:* If successive approximations are used and y_0 and M_0 are the vertical deflections and bending moments, respectively for the elastic theory,

Problem 10.14

the first trial values y_1 used in the deformation theory should be taken as

$$y_1 = \frac{y_0}{1 - Hy_0/M_0}.$$

References

1 N. M. Newmark, "Numerical Procedure for Computing Deflections, Moments, and Buckling Loads," *Trans. Am. Soc. C. E.*, Vol. 108 (1943).
2 D. B. Steinman, *Suspension Bridges*, John Wiley and Sons.
3 D. B. Steinman, "A Generalized Deflection Theory for Suspension Bridges," *Trans. Am. Soc. C. E.*, Vol. 100 (1935).
4 L. S. Moisseiff and F. Lienhard, "Suspension Bridges under the Action of Lateral Forces." *Trans. Am. Soc. C. E.*, Vol. 98 (1933).
5 S. Timoshenko, "The Stiffness of Suspension Bridges," *Trans. Am. Soc. C. E.*, Vol. 94 (1930).
6 "George Washington Bridge," *Trans. Am. Soc. C. E.*, Vol 94 (1930).
7 George C. Priester, "Applications of Trigonometric Series to Cable Stress Analysis in Suspension Bridges," *Eng. Research Bull.* 12, University of Michigan.
8 A. A. Jakkula, "The Theory of the Suspension Bridge," *Int. Assoc. for Bridge and Struct. Eng.*, Vol. IV (1936).
9 L. M. Legatski, "The Theory of the Continuous Suspension Bridge," Dissertation, University of Michigan, 1937.
10 Shortridge Hardesty and Harold E. Wessman, "Preliminary Design of Suspension Bridges," *Trans. Am. Soc. C. E.*, Vol. 104 (1939).
11 D. B. Steinman "Aerodynamic Theory of Bridge Oscillations," *Trans. Am. Soc. C. E.*, Vol. 115 (1950).
12 "Aerodynamic Stability of Suspension Bridges," *Trans. Am. Soc. C. E.*, Vol. 120 (1955).
13 David J. Peery "An Influence-Line Analysis for Suspension Bridges," *Trans. Am. Soc. C. E.*, Vol. 121 (1956).
14 Robert S. Rowe "Amplified Stress and Displacement in Guyed Towers," *Trans. Am. Soc. C. E.*, Vol. 125 (1960).
15 E. Cohen and H. Perrin "Design of Multi-Level Guyed Towers," *Proc. Am. Soc. C. E.*, Vol. 83, September 1957.
16 Donald L. Dean, "Static and Dynamic Analysis of Guy Cables," *Trans. Am. Soc. C. E.*, Vol. 127, Part II (1962).
17 A. Freudenthal, "Deflection Theory for Arches," *Publ. Int. Assoc. for Bridge and Struct. Eng.*, Vol. III (1935).
18 "Rainbow Arch Bridge over Niagara Gorge," *Trans. Am. Soc. C. E.*, Vol. 110 (1945).
19 S. O. Asplund "Deflection Theory of Arches," *Proc. Am. Soc. C. E.*, October 1961.
20 M. G. Kaldjian "Prestressed Bowstring Arch," *Proc. Am. Soc. C. E.*, October 1961.
21 S. Chandrangsu and S. R. Sparkes "A Study of the Bowstring Arch Having Extensible Suspension Rods and Different Ratios of Tie-Beam to Arch-Rib Stiffness," *Inst. of Civil Eng.*, Vol. 3, No. 2, August 1954.
22. F. Bleich, "Dynamic Instability of Truss-Stiffened Suspension Bridges under Wind Action," *Trans. Am. Soc. C. E.*, Vol. 114 (1949).
23 Robert S. Rowe, "Amplification of Stress in Flexible Steel Arches," *Trans. Am. Soc. C. E.*, Vol. 119 (1954).

11

Special Problems in Statically Indeterminate Stresses

FRAMES WITH SEMIRIGID CONNECTIONS

11.1 Introduction

In many types of structures the beams are riveted or welded to columns and girders by angle, plate, and tee sections. A typical example of a beam-to-column connection is illustrated in Fig. 11.1 in which an I-beam frames into an H-column. The flanges of the beam are connected to the flanges of the column by two tee sections, whereas the web is connected by angles. Angles are also commonly used to connect the flanges. The tee, plate, and angle sections are fastened to the beams and columns by riveted, bolted, or welded connections.

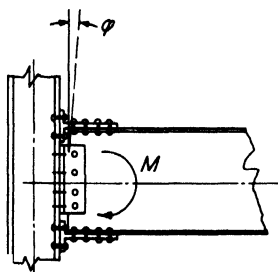



Fig. 11.1

Such structural details necessitate a discontinuity in the cross section of the beam that may increase noticeably the strain at the ends of the member. The magnitude of this strain in the connection details cannot be alculated accurately by rational methods, for local stress concentration nd deformation make the problem too complicated. Because the reactive forces between the column and the beam must pass through

the rivets, bolts, or weld, severe local deformation may occur in addition to the distortion of the tee and angle sections. In experimental work it is difficult to evaluate the separate effects of such individual factors as change in length of rivets, local bending of angles or tees, shearing deformation, and local bending of the column flanges. For this reason experimental results for a particular arrangement of connection details can seldom be applied to other types of connections, and consequently it is desirable to have experimental data for each type of connection that is used.

11.2 Experimental Results

Considerable experimental work has been performed in England and in the United States on the deformation of typical riveted and welded beam connections. The effect of this deformation on the stresses in beams and columns has also been studied. Particular mention will be made at this time of the results of a comprehensive research program that are recorded in the First, Second, and Final Reports of the Steel Structures Research Committee of Great Britain and to the tests conducted at Lehigh University for the American Institute of Steel Construction. In this research work many tests were made of the amount of deformation in beam connections similar to the one shown in Fig. 11.1. The deformation within the connection is measured by the relation between the moment applied to the connection and the rotation ϕ of the end of the beam with respect to the axis of the column. Typical curves recorded in the Second Report of the Steel Structures Research Committee are shown in Fig. 11.2. It is apparent from these curves that no linear relationship exists between the applied moment M and the end rotation ϕ of the beam connection. However, a somewhat rough approximation of the curve over a limited range can be made by a straight line, which, if the slope is selected somewhat low, will give a conservative design.

When the moment-rotation curve for a typical beam connection is approximated by one or several straight lines, the moment can be expressed by equations of the type

$$M = \psi\phi \tag{11.1}$$

where ψ is the slope of the straight line. The quantity M/ψ is equivalent to M/EI in a beam and can be treated as such in the calculations.

11.3 Slope-Deflection Equations for Semirigid Connections

Once the stiffness factor ψ of the connection is selected from tests it can be used in determining the coefficients for the slope deflection equations.

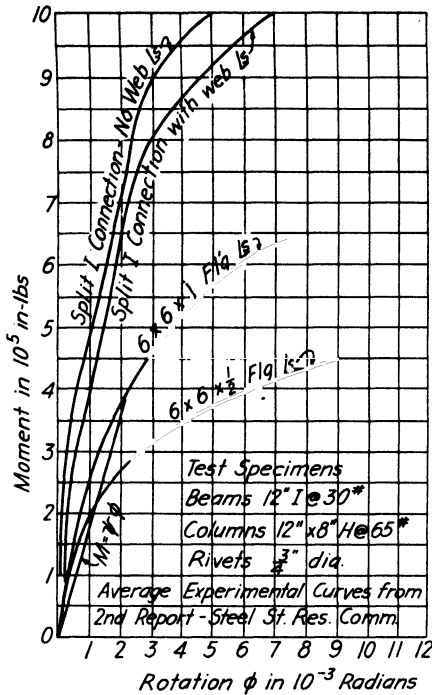



Fig. 11.2

In these calculations it is convenient to consider the quantity M/ψ as a concentrated load on the conjugate beam, which makes the connection equivalent to a change in cross section of the beam. The derivation can then proceed in the manner explained in Article 4.3 for beams of constant cross section, and Article 7.2 for variable cross section, by superimposing the rotations due to the two end moments. A unit moment at end a in member ab, Fig. 11.3a would give a load on the conjugate beam as shown, where ψ_a and ψ_b are the stiffness factors for the connections at a and b,

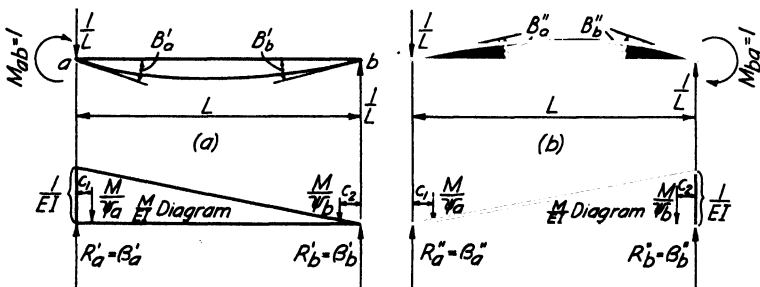



Fig. 11.3

respectively. The distances c_1 and c_2 are commonly assumed equal to zero, which is equivalent to placing the connection at the center of the joint.

For a unit moment applied at a with the distances c_1 and c_2 equal to zero, the end rotations are

$$\beta_a' = \frac{L}{3EI} + \frac{1}{\psi_a} \tag{11.2a}$$

$$\beta_b' = -\frac{L}{6EI} \tag{11.2b}$$

When a unit end moment is applied at b, Fig. 11.3b, the end rotations are

$$\beta_a'' = -\frac{L}{6EI} \tag{11.3a}$$

$$\beta_b'' = \frac{L}{3EI} + \frac{1}{\psi_b} \tag{11.3b}$$

The total rotations for any end moments are

$$\theta_a = M_{ab}\left(\frac{L}{3EI} + \frac{1}{\psi_a}\right) - \frac{M_{ba}L}{6EI} \tag{11.4a}$$

$$\theta_b = -\frac{M_{ab}L}{6EI} + M_{ba}\left(\frac{L}{3EI} + \frac{1}{\psi_b}\right) \tag{11.4b}$$

When equations 11.4a and b are solved for M_{ab} and M_{ba}, the usual form of the slope deflection equations is obtained, that is,

$$M_{ab} = \frac{EI}{L}(C_1\theta_a + C_2\theta_b) \tag{11.5a}$$

$$M_{ba} = \frac{EI}{L}(C_2\theta_a + C_3\theta_b) \tag{11.5b}$$

where

$$C_1 = \frac{12C''}{4C'C'' - 1} \tag{11.6a}$$

$$C_2 = \frac{6}{4C'C'' - 1} \tag{11.6b}$$

$$C_3 = \frac{12C'}{4C'C'' - 1} \tag{11.6c}$$

$$C' = 1 + \frac{3K}{\psi_a} \tag{11.6d}$$

$$C'' = 1 + \frac{3K}{\psi_b} \tag{11.6e}$$

$$K = \frac{EI}{L}$$

11.4 Fixed-end Moments for Semirigid Connections

The fixed-end moments for beams with constant cross section and with semirigid connections can be expressed in terms of the fixed-end moments for rigid connections and the coefficients C_1, C_2, C_3. Let M_{Fab} and M_{Fba} denote the fixed-end moments for rigid connections, and M'_{Fab} and M'_{Fba} the corresponding values for semirigid connections, when c_1 and c_2 in Fig. 11.3 are neglected. These connections are therefore assumed to be at the center of the joints. The values of M'_{Fab} can be obtained from the following equations:

$$M'_{Fab} = \tfrac{1}{6}[M_{Fab}(2C_1 - C_2) + M_{Fba}(2C_2 - C_1)] \qquad (11.7a)$$

$$M'_{Fba} = \tfrac{1}{6}[M_{Fab}(2C_2 - C_3) + M_{Fba}(2C_3 - C_2)] \qquad (11.7b)$$

where C_1, C_2, and C_3 have the values given by equations 11.6*a*, *b*, and *c*.

These equations can be derived in the following manner. First calculate the end rotations α for a simply supported beam in terms of the ordinary fixed-end moments. The magnitude of these end rotations are

$$\alpha_a = -\frac{M_{Fab}}{3K} + \frac{M_{Fba}}{6K}$$

$$\alpha_b = \frac{M_{Fab}}{6K} - \frac{M_{Fba}}{3K}$$

To determine the end couples M_F' necessary to reduce these rotations to zero, equations 11.5*a* and *b* can be used. From equation 11.5*a*

$$M'_{Fab} = K[C_1(-\alpha_a) + C_2(-\alpha_b)]$$

or

$$M'_{Fab} = K\left[C_1\left(\frac{M_{Fab}}{3K} - \frac{M_{Fba}}{6K}\right) + C_2\left(-\frac{M_{Fab}}{6K} + \frac{M_{Fba}}{3K}\right)\right]$$

giving

$$M'_{Fab} = \tfrac{1}{6}[M_{Fab}(2C_1 - C_2) + M_{Fba}(2C_2 - C_1)]$$

The value of M'_{Fba} can be obtained from equation 11.5*b* in a similar manner.

For a uniformly distributed load w across the entire span, the values of the fixed-end moments are

$$M'_{Fab} = -\frac{wL^2}{24}(C_1 - C_2) \qquad (11.8a)$$

$$M'_{Fab} = \frac{wL^2}{24}(C_3 - C_2) \qquad (11.8b)$$

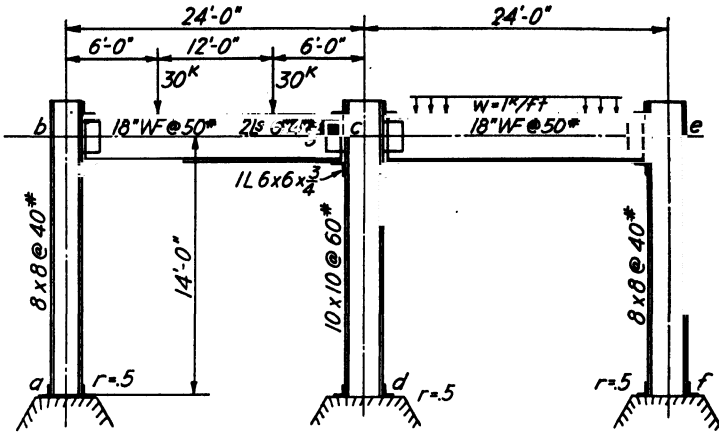



Fig. 11.4

Once the values of the ψ terms are selected, the analysis of any structural frame is made in the same manner as for members with variable moments of inertia.

Example 11.1 The frame in Fig. 11.4 will be analyzed for semirigid connections between the beams and columns and for a condition of half fixity at the base of the columns. From a study of test results the value of ψ for all connections will be selected as

$$\psi = 100 \times 10^6 \text{ in.-lb per radian}$$

Therefore we obtain for the 18-in. WF at 50-lb beams from equations 11.6*d* and *e*

$$C' = C'' = 1 + \frac{(3)(30)(10^6)(800)}{(24)(12)(100)(10^6)} = 3.50$$

From equations 11.6*a*, *b*, and *c*

$$C_1 = C_3 = \frac{(12)(3.50)}{(4)(3.50)^2 - 1} = 0.875$$

$$C_2 = \tfrac{6}{48} = 0.125$$

For span *bc*,

$$M_{Fbc} = -\frac{(30)(6)(18)^2}{(24)^2} - \frac{(30)(18)(6)^2}{(24)^2} = -135 \text{ ft-kips}$$

$$M_{Fcb} = +135 \text{ ft-kips}$$

From equations 11.7*a* and *b*

$$M'_{Fbc} = \tfrac{1}{6}[-135(1.75 - 0.125) + 135.0(0.25 - 0.875)] = -50.63 \text{ ft-kips}$$

$$M'_{Fcb} = +50.63 \text{ ft-kips}$$

From equations 11.8*a* and *b*

$$M'_{Fce} = -\frac{(1.0)(24)(24)}{(24)}(0.875 - 0.125) = -18.0 \text{ ft-kips}$$

$$M'_{Fec} = +18.0 \text{ ft-kips}$$

Table 11.1 Distribution and Carry-Over Factors

Member	C	$K = \frac{I}{L}$	CK	r	Carry-Over
			Joint b		
ba	3.5	0.871	3.05	0.556	$\frac{2(1-0.5)}{4-0.5} = 0.286$
bc	0.875	2.78	2.44	0.444	$\frac{0.125}{0.875} = 0.143$
			$\Sigma CK = 5.49$	1.000	
			Joint c		
cb	0.875	2.78	2.44	0.203	0.143
ce	0.875	2.78	2.44	0.203	0.143
cd	3.5	2.04	7.15	0.594	0.286
			$\Sigma CK = 12.03$	1.00	

The distribution of the fixed-end moments is shown in Fig. 11.5*a*. The numerical calculations are the same as for beams with rigid connections. From the end moments in the columns, the horizontal restraining force at b is found to be

$$F = 1.90 + 0.87 - 2.64 = 0.13 \text{ kip}$$

This restraining force is removed in the usual manner by assuming that points b, c, and e have some horizontal displacement, say

$$E\Delta = 100 \text{ units}$$

Therefore

$$M_{Fab} = M_{Fba} = M_{Fef} = M_{Ffe} = \frac{(6)(0.871)(100)}{14} = 37.4$$

$$M_{Fcd} = M_{Fdc} = \frac{(6)(2.04)(100)}{14} = 87.3$$

The distribution of these fixed-end moments is shown in Fig. 11.5*b*. It should be noticed that a condition of half fixity is maintained at the base of all columns by first distributing the fixed-end moment at the base,

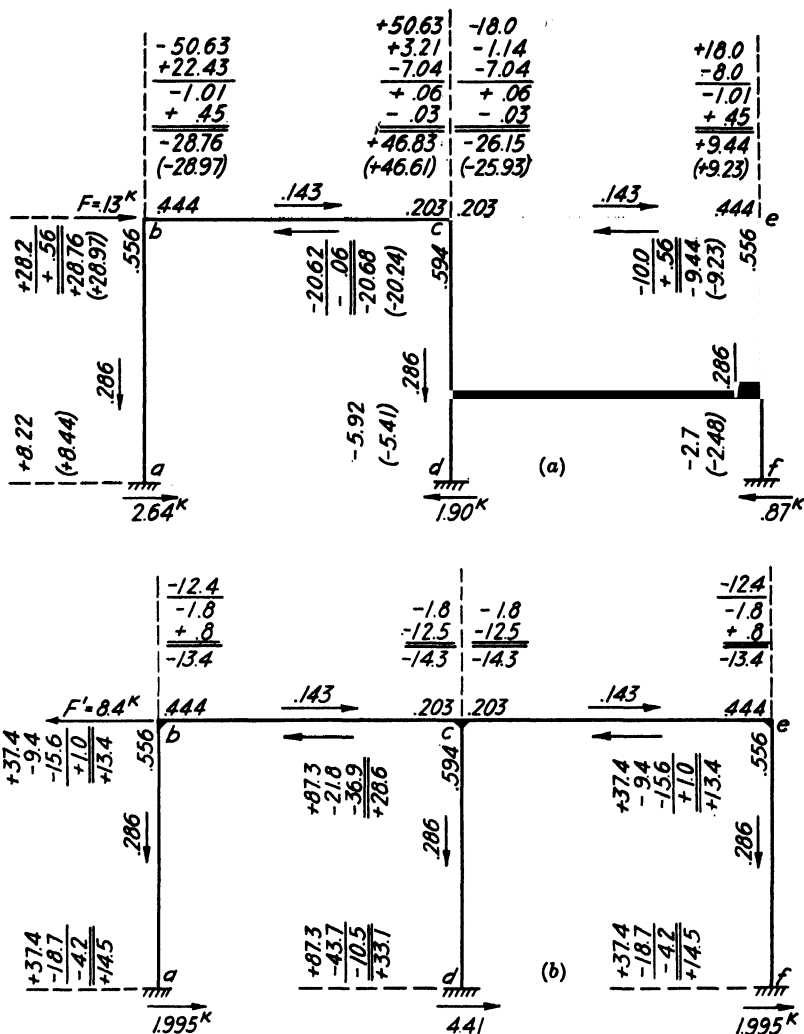



Fig. 11.5

then using a coefficient of 3.5 in calculating the distribution factor at the top of the column and finally a carry-over factor of 0.286 back to the base. The force F' necessary to give the final end moments is

$$F' = 1.995 + 4.41 + 1.995 = 8.40 \text{ kips}$$

Therefore the correct moments are obtained by adding, to the moments in Fig. 11.5*a*, 0.13/8.40 times the moments in Fig. 11.5*b*. The final values are given in parenthesis in Fig. 11.5*a*.

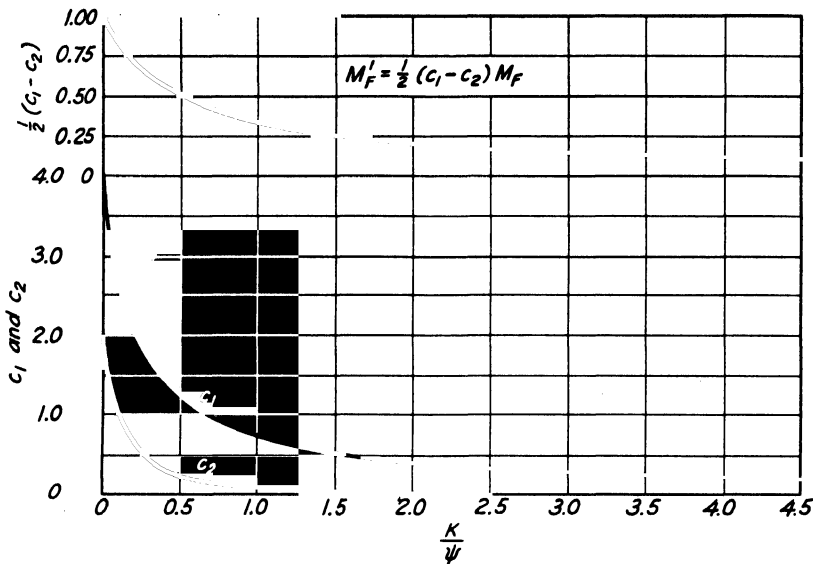



Fig. 11.6

11.5 *Discussion of Semirigid Connections*

Since the magnitudes of the coefficients C_1, C_2, and the fixed-end moments M'_{Fab} and M'_{Fba} are important factors in the structural design, it is worthwhile to note, as illustrated in Fig. 11.6, the effect on them of varying the K/ψ value. It is apparent that a particular type of end connection may provide considerable end restraint for one beam but relatively little for another, for the larger beams will require more rigid connections for the same degree of end restraint.

The rapid decrease in the values of C_1, C_2, and $(C_1 - C_2)/2$ in Fig. 11.6 for relatively small increases in K/ψ, particularly in the region from 0 to 0.5, is a disturbing feature of this diagram. This large variation is probably one of the reasons for the reluctance of engineers to assume approximate values for semirigid connections, particularly for design purposes. When results of laboratory tests of specific end connections are not available, it will be necessary to fabricate and test several beam and column specimens. Information on laboratory procedures and discussion of test data can be found in the references listed at the end of the chapter. Probably the simplest laboratory procedure is the one presented by the author in Ref. 6, for this procedure involves measuring vertical deflection only.

CALCULATION OF STRESSES IN SPACE FRAMES

11.6 Deformation Equations for Axial Stress

In Chapter 4, equations were derived that expressed the value of the end moments acting on any member in terms of the rotation and translation of the ends of the members. These deformation equations, or slope deflection equations as they were called, were found to be extremely useful in the solution of statically indeterminate frame structures. Similar equations will now be developed for the axial stress in any member in a space frame in terms of the linear motion of the ends of the members.

Let the member ab (Fig. 11.7) represent any member in a space structure whose ends a and b have coordinates x_a, y_a, z_a, and x_b, y_b, z_b with respect to some origin O. The linear movement of the end points a and b in the x, y, z directions will be designated by u_a, v_a, w_a and u_b, v_b, w_b, respectively. The projections of the length ab on the three coordinate axes will be called X_{ab}, Y_{ab}, and Z_{ab}, so that

$$X_{ab} = x_a - x_b$$

$$Y_{ab} = y_a - y_b$$

$$Z_{ab} = z_a - z_b$$

In terms of orthogonal projections, the length of the member L is defined by the relation

$$L^2 = X^2 + Y^2 + Z^2 \qquad (11.9a)$$

and, by the usual rules of variation, we obtain the relation

$$2L(\Delta L) = 2X(\Delta X) + 2Y(\Delta Y) + 2Z(\Delta Z)$$

Therefore

$$\Delta L = \frac{1}{L}\,[(u_a - u_b)X + (v_a - v_b)Y + (w_a - w_b)Z] \qquad (11.9b)$$

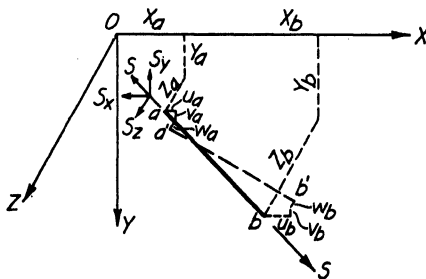



Fig. 11.7

since

$$\Delta X = u_a - u_b$$
$$\Delta Y = v_a - v_b$$
$$\Delta Z = w_a - w_b$$

But for any axial stress S in the member,

$$\Delta L = \frac{SL}{AE}$$

in which AE equals the cross-sectional area times the modulus of elasticity. Therefore

$$S = \frac{AE}{L}\Delta L = \frac{AE}{L^2}[(u_a - u_b)X + (v_a - v_b)Y + (w_a - w_b)Z] \quad (11.10)$$

The components of the stress S in the direction of the selected axes will therefore be equal to

$$S_x = \frac{SX}{L} = \frac{AE}{L^3}[(u_a - u_b)X^2 + (v_a - v_b)XY + (w_a - w_b)XZ] \quad (11.11a)$$

$$S_y = \frac{SY}{L} = \frac{AE}{L^3}[(u_a - u_b)XY + (v_a - v_b)Y^2 + (w_a - w_b)YZ] \quad (11.11b)$$

$$S_z = \frac{SZ}{L} = \frac{AE}{L^3}[(u_a - u_b)XZ + (v_a - v_b)YZ + (w_a - w_b)Z^2] \quad (11.11c)$$

If

$$Q = \frac{AE}{L^3}$$

the components of stress for any member can be expressed in terms of the linear displacements of the joints by equations 11.11*a*, *b*, and *c* as soon as the quantities QX^2, QY^2, QZ^2, QXY, QXZ, and QYZ are determined.

11.7 Equilibrium and Compatibility Equations

Since the conditions of statical equilibrium at any joint require that there be no unbalanced components at that point, the following equilibrium equations must be satisfied at each joint:

$$\sum S_x - \sum F_x = 0 \quad (11.12a)$$
$$\sum S_y - \sum F_y = 0 \quad (11.12b)$$
$$\sum S_z - \sum F_z = 0 \quad (11.12c)$$

In these equations, F_x, F_y, and F_z represent the components of the external loads applied at the joint.

If the values of S_x, S_y, S_z from equations 11.11 are substituted in the equilibrium equations 11.12, the following compatibility equations for any joint a are obtained:

$$u_a \sum QX^2 + v_a \sum QXY + w_a \sum QXZ - \sum u_n(QX^2)_n - \sum v_n(QXY)_n - \sum w_n(QXZ)_n - \sum F_{xa} = 0 \quad (11.13a)$$

$$u_a \sum QXY + v_a \sum QY^2 + w_a \sum QYZ - \sum u_n(QXY)_n - \sum v_n(QY^2)_n - \sum w_n(QYZ)_n - \sum F_{ya} = 0 \quad (11.13b)$$

$$u_a \sum QXZ + v_a \sum QYZ + w_a \sum QZ^2 - \sum u_n(QXZ)_n - \sum v_n(QYZ)_n - \sum w_n(QZ^2)_n - \sum F_{za} = 0 \quad (11.13c)$$

In equations 11.13 the subscript n refers to any joint connected to the joint a by a member. Until the reader is familiar with the procedure, it will probably be better for him to substitute the expressions for S_x, S_y, and S_z from equations 11.11 directly into equations 11.12. Once the notation is thoroughly understood, however, equations 11.13 save considerable time.

11.8 Use of the Deformation Equations

Calculating the stresses in space frames by means of equations 11.11 and 11.12 is similar to solving for end moments by the slope deflection method. The final equations 11.13 have the linear displacements of the joints as unknowns, and, as there are three such equations for each joint, a unique solution is possible. After the displacements u, v, and w of each joint have been determined, the stress in each member is known from equation 11.10. This method of analysis requires the solving of $3n$ simultaneous equations, where n is the number of joints free to move in any direction. The solution of this number of equations can become laborious for a structure with many joints. The use of group or block displacements as suggested by Southwell (Ref. 7) to simplify the mathematical procedure will be discussed later.

Considerable care must be exercised to maintain a consistent sign convention throughout the calculations. First, the coordinates x, y, and z of each joint must be established correctly with respect to the reference axes. Then the projections X, Y, and Z for each member must be made consistent with the coordinates of the joints by subtracting the coordinates for the opposite end of each member from the coordinates of the joint for which the equations are to be written. That is, if a member ab has coordinates (6, −4, −8) for point a and (−4, 6, −20) for point b, the

values of X, Y, and Z to be used in expressing the components of stress at point a are

$$X_{ab} = x_a - x_b = 6 - (-4) = 10$$
$$Y_{ab} = y_a - y_b = (-4) - (6) = -10$$
$$Z_{ab} = z_a - z_b = -8 - (-20) = +12$$

For writing the necessary equations for the stress components at point b, the values are

$$X_{ba} = -4 - (6) = -10$$
$$Y_{ba} = 6 - (-4) = 10$$
$$Z_{ba} = -20 - (-8) = -12$$

That is, the signs for point b are opposite to those for point a.

If the proper signs for X, Y, and Z are substituted into the equations, S_{xa}, S_{ya}, S_{za}, the components acting upon the member ab at end a, will have directions consistent with the directions of the references axes x, y, and z. The stress S in the member ab as obtained from equation 11.10 will be tension when positive and compression when negative. It is therefore apparent that the correct signs for all stresses are obtained automatically if the quantities X, Y, and Z are consistent with the coordinates of the joints.

Example 11.2 The stresses for all members of the space frame in Fig. 11.8 will be determined by the deformation equations. Since points A, B, C, and D are considered fixed supports, only the displacements of joints E, F, and G are involved in the analysis. The necessary constants in terms of Q, X, Y, and Z are tabulated in Table 11.2 for these joints.

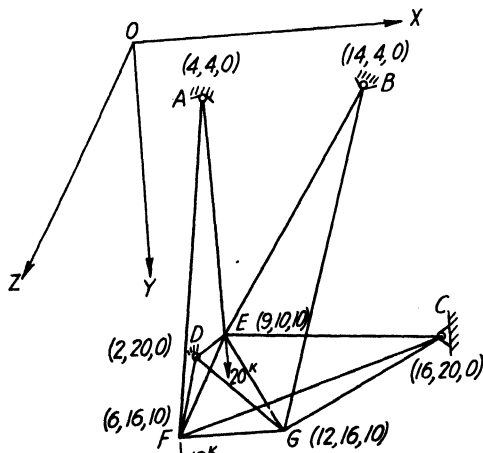



Fig. 11.8

The quantities from this table are then substituted into the compatibility equations 11.13*a*, *b*, and *c*, which are written for each joint *E*, *F*, and *G*. This operation gives equations 11.14 which are solved for the displacements *u*, *v*, and *w* for each joint. Various methods of solving such simultaneous equations are possible, but, as each person has his favorite

Table 11.2

Member	*A*, in.²	*L*, ft	$Q = \frac{A}{L_3} \times 10^{-3}$	QX^2	QY^2	QZ^2	QXY	QXZ	QYZ
EA	5	12.68	0.245	6.13	8.81	24.5	7.35	12.25	14.70
EB	5	12.68	0.245	6.13	8.81	24.5	−7.35	−12.25	14.70
EC	8	15.78	0.204	10.0	20.4	20.4	14.28	−14.28	−20.4
ED	8	15.78	0.204	10.0	20.4	20.4	−14.28	14.28	−20.4
EF	3	6.70	0.997	8.98	35.95	0	−17.97	0	0
EG	3	6.70	0.997	8.98	35.95	0	17.97	0	0
			Total	50.22	130.32	89.8	0	0	−11.4
FA	5	15.76	0.128	0.51	18.41	12.80	3.07	2.56	15.35
FC	3	14.7	0.095	9.46	1.51	9.46	3.79	−9.46	−3.79
FD	3	11.49	0.198	3.17	3.17	19.80	−3.17	7.92	−7.92
FE	3	6.7	0.997	8.98	35.95	0	−17.97	0	0
FG	3	6.0	1.39	50.0	0	0	0	0	0
			Total	72.12	59.04	42.06	−14.28	1.02	3.64
GB	5	15.76	0.128	0.51	18.41	12.80	−3.07	−2.56	15.35
GC	3	11.49	0.198	3.17	3.17	19.80	3.17	−7.92	−7.92
GD	3	14.7	0.095	9.46	1.51	9.46	−3.79	9.46	−3.79
GE	3	6.7	0.997	8.98	35.95	0	17.97	0	0
GF	3	6.0	1.39	50.0	0	0	0	0	0
			Total	72.12	59.04	42.06	14.28	−1.02	3.64

JOINT *E*

$$50.22u_e - 8.98u_f - 8.98u_g + 17.97v_f - 17.97v_g = 0$$
$$130.3v_e - 11.4w_e + 17.97u_f - 17.97u_g - 35.95v_f - 35.95v_g = 20{,}000 \text{ lb}$$
$$-11.4v_e + 89.8w_e = 0 \text{ or } w_e = 0.127v_e$$

JOINT *F*

$$72.12u_f - 14.28v_f + 1.02w_f - 8.98u_e - 50.0u_g + 17.97v_e = 0$$
$$-14.28u_f + 59.04v_f + 3.64w_f + 17.97u_e - 35.95v_e = 12{,}000 \text{ lb}$$
$$1.02u_f + 3.64v_f + 42.06w_f = 0 \text{ or } w_f = -0.024u_f - 0.0865v_f$$

JOINT *G*

$$72.12u_g + 14.28v_g - 1.02w_g - 8.98u_e - 50.0u_f - 17.97v_e = 0$$
$$14.28u_g + 59.04v_g + 3.64w_g - 17.97u_e - 35.95v_e = 0$$
$$-1.02u_g + 3.64v_f + 42.06w_g = 0 \text{ or } w_g = 0.024u_g - 0.0865v_g \qquad (11.14)$$

procedure, no particular solution will be given here.

Solving these equations, we obtain

$$u_e = -81.1 \qquad u_f = 39.5 \qquad u_g = 66.4$$
$$v_e = 323.8 \qquad v_f = 436.1 \qquad v_g = 156.3$$
$$w_e = 41.1 \qquad w_f = -38.7 \qquad w_g = -11.9$$

Substituting these values back in equations 11.10 and 11.11 gives the stresses shown in Table 11.3.

Table 11.3

Member	S	L	S_x	S_y	S_z
EA	+6,040	12.68	+2,390	+2,860	+4,770
EB	+8,550	12.68	−3,380	+4,050	+6,750
EC	−7,250	15.78	+3,220	+4,600	−4,600
ED	−10,920	15.78	−4,850	+6,920	−6,920
EF	+2,090	6.70	+940	−1,870	0
EG	−3,760	6.70	+1,690	+3,370	0
		Total	+10	+19,930	0
FA	+9,920	15.76	+1,260	+7,560	+6,300
FC	−3,520	14.7	+2,400	+960	−2,400
FD	−4,480	11.49	−1,570	+1,570	−3.910
FE	+2,090	6.7	−940	+1,870	0
FG	+1,348	6.0	−1,350	0	0
		Total	−200	+11,960	−10
GB	+3,280	15.76	−420	+2,500	+2,080
GC	−2,040	11.49	+710	+710	−1,780
GD	−162	14.7	−110	+40	−110
GE	−3,760	6.7	−1,690	−3,360	0
GF	+1,348	6.0	+1,350	0	0
		Total	−160	−110	+190

The numerical values of the stress given in Table 11.3 are then obtained by substituting the numerical value of the displacements into equation 11.10. The components S_x, S_y, and S_z are obtained by multiplying the stress by the ratios X/L, Y/L, and Z/L. The discrepancies in the summations of the X, Y, and Z components are due to solution of the simultaneous equations by successive approximations. However, the results are sufficiently accurate for all practical purposes.

11.9 Group and Rigid Block Displacements

In the preceding discussion and application of the deformation equations it was assumed that the movement of the joints was governed only by equations 11.13*a*, *b*, and *c*. However, the conditions of the problem will frequently permit a simplification of the equations by assuming relations between the relative movement of the joints. Such assumptions must, of course, be based on an estimation of the actual physical conditions that exist in the structure and will probably never provide an exact solution. Nevertheless, such assumptions will usually be necessary to

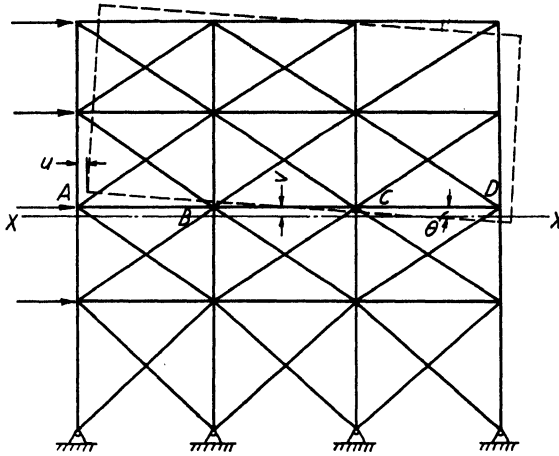



Fig. 11.9

make the solution possible from a practical viewpoint, and the results will, in general, satisfy the equilibrium conditions. Any local discrepancy can be distributed later without introducing any serious errors.

The assumption of rigid block displacements will be illustrated by the solution of the structure in Fig. 11.9. An exact analysis for all stresses by considering the movement of every joint would be impractical. However, if the portion above section x-x is assumed to move as a rigid body, the movement of points A, B, C, and D can be expressed in terms of one vertical displacement v, a horizontal displacement u, and a rotation θ. These displacements must satisfy the three equilibrium conditions for the rigid body, and consequently the problem reduces to three equations with three unknowns. Other sections can then be taken and various sets of stresses obtained that satisfy the equilibrium condition for the structure as a unit. Local discrepancies in the equilibrium conditions at a joint, or in the strain relations, must be corrected by approximation. The success of such an analysis will depend upon the astuteness of the person making the calculations, for the procedure must be varied to fit the conditions of the problem.

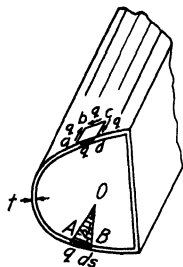
Instead of relating the movements of the various joints by assuming a rigid body condition, it will sometimes be better to assume other types of related displacements which can be combined to satisfy the compatibility equations. Such solutions will usually be involved, but are often the most direct and accurate method of analysis for complicated space frameworks. The reader should consult the references at the end of the chapter, particularly Ref. 8, for applications of this method to such problems as the hull stresses in rigid airships.

SHEARING STRESSES IN THIN-WALLED CLOSED SECTIONS

11.10 Shearing Stress due to Torsion

The shearing stress f_s on any element ds of a thin-walled closed section (Fig. 11.10) produced by a twisting moment T is assumed to be uniformly distributed over the thickness t in the plane of the cross section. The shearing force q for ds equal to unity is therefore equal to

Fig. 11.10

$$q = f_s t \text{ lb per linear inch} \qquad (11.15)$$

where q is defined as the unit shearing force and the distribution of q around the perimeter of a closed section is termed the shear flow. The value of q in a single cell closed section subjected to torsion only must be constant around the section regardless of the thickness t. This statement follows from the fact that the unit shearing force on an element such as *abcd* in Fig. 11.10 can vary only if there is a change in the normal stress or if some external shearing stress is applied to the surface. Since these axial or external stresses are not acting when the section is subjected to torsion only, the shearing force q must be constant for all elements.

The resisting torque about any point O due to any element whose length is ds is

$$dT = (q\,ds)h = 2q\,(\text{area } OAB) \qquad (11.16)$$

The total resisting torque T_r which must be equal to the applied torque T is therefore

$$T_r = T = \oint qh\,ds = 2qA$$

where A is the total area enclosed by the section and $\oint$ represents integration around the perimeter. The constant unit shearing force q is therefore easily calculated by the following expression, usually called Bredt's formula,

$$q = \frac{T}{2A} \qquad (11.17)$$

11.11 Shearing Deformation due to Torsion

If a cut is made at any section x-x in a closed section (Fig. 11.11), the shearing deformation of all elements will tend to produce a relative movement of the two sides. The effect of the shearing stress on one

element is shown in Fig. 11.11. The movement of points a and b for a unit of length ad and unit perimeter ab is

$$\delta_s = \frac{f_s}{G} = \frac{q}{tG} \tag{11.18}$$

The internal work or strain energy in the material for any distance ds along the perimeter is therefore

$$dW_i = \frac{q}{2}\delta_s\, ds = \frac{Tq\, ds}{4AtG}$$

If the torque about any fixed point O is T, it can be replaced by the couple

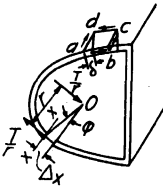
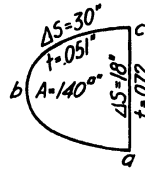



Fig. 11.11 *Fig. 11.12*

T/r as shown in Fig. 11.11. The external work per unit of length is therefore equal to

$$\frac{T}{2r}\Delta_x = \frac{T\phi}{2}$$

Equating the external work to the total internal work per unit length of cell, we obtain

$$\frac{T\phi}{2} = \frac{T}{4AG}\oint \frac{q\, ds}{t}$$

or

$$\phi = \frac{1}{2AG}\oint \frac{q\, ds}{t} \tag{11.19a}$$

where ϕ is the angle of twist per unit length of cell.

In general, the perimeter is divided into several elements whose lengths are known and over which the shear flow q and thickness t are constant. The integration should then be replaced by the summation of the terms for the elements, that is,

$$\phi = \frac{1}{2AG}\sum \frac{q\,\Delta s}{t} \tag{11.19b}$$

Example 11.3 The shearing stresses and the angle of twist will be calculated for the cell in Fig. 11.12 for a torque of 30,000 in.-lb.

$$q = \frac{30{,}000}{(2)(140)} = 107 \text{ lb per inch}$$

For nose section

$$f_s = \frac{107}{0.051} = 2100 \text{ lb per square inch}$$

For web

$$f_s = \frac{107}{0.072} = 1490 \text{ lb per square inch}$$

From equation 11.19*b*,

$$\phi = \frac{107}{(2)(140)G}\left(\frac{30}{0.051} + \frac{18}{0.072}\right) = \frac{320}{G} \text{ radians per inch of length}$$

11.12 Torsional Stresses in Multiple-Cell Sections

The shearing stresses in the walls of multiple-cell sections such as Fig. 11.13 are calculated by means of equations 11.17 and 11.19 together with the following conditions.

1. The shear flow q' into an intersection of elements must equal the shear flow out or $q' - q'' - q''' = 0$ (see Fig. 11.13*a*).
2. The angle of twist ϕ for each cell is the same.
3. The sum of the resisting torques for all cells must equal the total torque applied to the section.

Condition 1 is based on the equilibrium of a corner element with respect to the axial direction. Taking

$$\sum F_x = 0$$

$$q' - q'' - q''' = 0$$

or

$$q''' = q' - q'' \tag{11.20}$$

The second condition 2 can be visualized by considering the movement of a gap in the element common to two cells with respect to the same point *O*. By referring to Figs. 11.11 and 11.13 it is apparent that

$$\phi_1 = \phi_2$$

and

$$\phi_2 = \phi_3$$

$$\phi_1 = \frac{1}{2A_1G}\sum\frac{q\,\Delta s}{t}$$

$$\phi_2 = \frac{1}{2A_2G}\sum\frac{q\,\Delta s}{t}$$

$$\phi_3 = \frac{1}{2A_3G}\sum\frac{q\,\Delta s}{t} \tag{11.21}$$

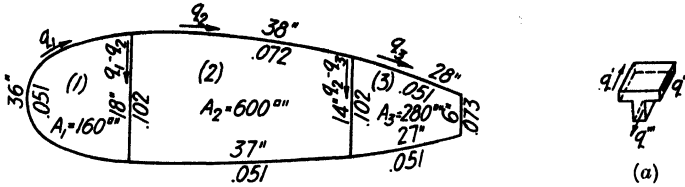



Fig. 11.13

The third condition 3 states that

$$T_1 + T_2 + T_3 = T$$

or

$$2(q_1A_1 + q_2A_2 + q_3A_3) = T \tag{11.22}$$

As the shear flow can be expressed in terms of one unknown q value for each cell, these relations are sufficient to obtain a unique solution.

Example 11.4 The shearing stresses in the three-cell section shown in Fig. 11.13 will be calculated for a torque of 40,000 ft-lb. The notation for the shear flow in the various elements is indicated on the sketch. By means of equation 11.19*b* the angle of twist ϕ for each cell can be written in the form

$$\phi_1 = \frac{1}{(2)(160)G}\left[\frac{36q_1}{0.051} + \frac{18(q_1 - q_2)}{0.102}\right]$$

$$\phi_2 = \frac{1}{(2)(600)G}\left[\frac{38q_2}{0.072} + \frac{14(q_2 - q_3)}{0.102} + \frac{37q_2}{0.051} + \frac{18(-q_1 + q_2)}{0.102}\right]$$

$$\phi_3 = \frac{1}{(2)(280)G}\left[\frac{28q_3}{0.051} + \frac{6q_3}{0.072} + \frac{27q_3}{0.051} + \frac{14(-q_2 + q_3)}{0.102}\right]$$

Equating

$$\phi_1 = \phi_2$$

and

$$\phi_2 = \phi_3$$

we obtain

$$2.76q_1 - 0.552q_2 = -0.147q_1 + 1.305q_2 - 0.114q_3$$

$$-0.147q_1 + 1.305q_2 - 0.114q_3 = -0.245q_2 + 2.31q_3$$

Eliminating q_3 from the preceding equations gives

$$q_2 = 1.625q_1$$

Eliminating q_2 from these equations gives

$$q_3 = 0.977q_1$$

Substituting these values of q_2 and q_3 into equation 11.22 gives

$$2[160q_1 + 600(1.625q_1) + 280(0.977q_1)] = (40{,}000)(12),$$

from which $q_1 = 170.5$ lb per inch

and therefore $q_2 = (1.625)(170.5) = 277$ lb per inch

$q_3 = (0.977)(170.5) = 166.5$ lb per inch

11.13 Flexural Stresses in Unsymmetrical Closed Sections

The normal stresses on the effective area of an unsymmetrical closed section, as in Fig. 11.14, can be calculated by the usual flexure formula if the principal axes of inertia are used as the axes of rotation. The principal axes of inertia OX' and OY' are ordinarily determined with respect to any reference axes OX and OY passing through the centroid O of the effective area by the equation

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} \tag{11.23}$$

The principal moments of inertia are then calculated by equations 11.24*a* and *b*.

$$I_{x'} = I_x \cos^2 \theta + I_y \sin^2 \theta - I_{xy} \sin 2\theta \tag{11.24a}$$

$$I_{y'} = I_y \cos^2 \theta + I_x \sin^2 \theta + I_{xy} \sin 2\theta \tag{11.24b}$$

in which I_x and I_y are the moments of inertia about OX and OY, respectively. I_{xy} is the product of inertia about axes OX and OY.

The other necessary relations are

$$M_{x'} = M_x \cos \theta - M_y \sin \theta \tag{11.25a}$$

$$M_{y'} = M_y \cos \theta + M_x \sin \theta \tag{11.25b}$$

$$x' = x \cos \theta + y \sin \theta \tag{11.25c}$$

$$y' = y \cos \theta - x \sin \theta \tag{11.25d}$$

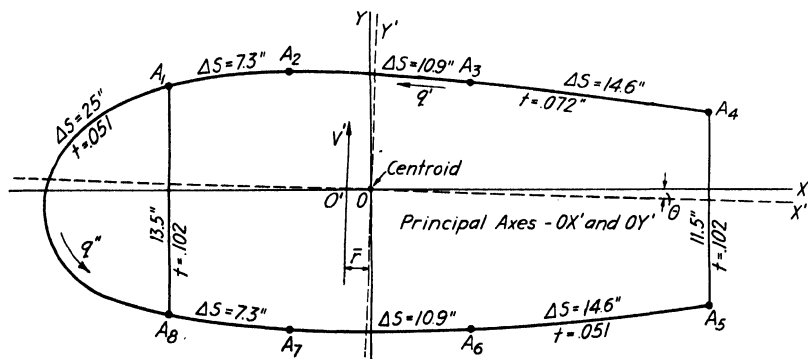



Fig. 11.14

The normal stresses f are calculated for bending about the principal axes by the usual bending formula

$$f = \frac{M_{x'}y'}{I_{x'}} + \frac{M_{y'}x'}{I_{y'}} \tag{11.26}$$

However, the normal stress can also be calculated directly from the reference axes OX and OY by the following equation (see Ref. 10):

$$f = \frac{(M_y I_x - M_x I_{xy})x}{I_x I_y - I_{xy}^2} + \frac{(M_x I_y - M_y I_{xy})y}{I_x I_y - I_{xy}^2} \tag{11.27}$$

Table 11.4 Properties with Respect to Axes OX and OY

No.	Area, in.^{2}	x	y	Ax^2	Ay^2	Axy
A_1	1.2	−12.1	+6.11	175.69	44.80	−88.72
A_2	0.6	−4.9	+6.81	14.41	27.83	−20.02
A_3	0.4	+5.9	+6.41	13.92	16.44	15.13
A_4	0.6	+20.3	+4.51	247.25	12.20	54.93
A_5	0.6	+20.3	−6.99	247.25	29.32	−85.14
A_6	0.4	+5.9	−8.29	13.92	27.49	−19.56
A_7	0.4	−4.9	−8.19	9.60	26.83	16.05
A_8	0.8	−12.1	−7.39	117.13	43.69	71.54
				I_y = 839.17	I_x = 228.60	I_{xy} = −55.79

Example 11.5 The normal stresses acting on the flange area A_1 in the closed section of Fig. 11.14 will be calculated by equations 11.24, 11.25, and 11.26. The properties of the section with respect to the axes OX and OY, where O is the centroid of the areas $A_1, A_2, A_3 \ldots$, are given in Table 11.4.

From equation 11.23 the angle θ that the principal axis OX' makes with OX is

$$\tan 2\theta = \frac{(2)(-55.79)}{839.17 - 228.60} = -0.1827$$

$$2\theta = 169° \, 38' \, 40'' \quad \text{or} \quad 349° \, 38' \, 40''$$

$$\theta = 84° \, 49' \, 20'' \quad \text{or} \quad -5° \, 10' \, 40''$$

$$\cos -5° \, 10' \, 40'' = 0.9959 \qquad \sin -5° \, 10' \, 40'' = -0.0901$$

$$x' = 0.9959x - 0.0901y$$

$$y' = 0.9959y + 0.0901x$$

From equations 11.24*a* and *b*, the values of the principal moments of inertia are

$$I_{x'} = (228.60)(0.9959)^2 + (839.17)(0.0901)^2 - (-55.79)(-0.1764)$$
$$= 223.70 \text{ in.}^4$$

$$I_{y'} = (839.17)(0.9959)^2 + (228.60)(0.0901)^2 + (-55.79)(-0.1764)$$
$$= 844.00 \text{ in.}^4$$

For area A_1

$$x' = (-12.1)(0.9959) + (6.11)(-0.0901) = -12.60$$
$$y' = (6.11)(0.9959) - (-12.1)(-0.0901) = 4.99$$

From equation 11.26

$$f = \frac{M_{x'}(4.99)}{223.7} + \frac{M_{y'}(-12.60)}{844.0} = 0.0223M_{x'} - 0.0149M_{y'}$$

but

$$M_{x'} = 0.9959M_x + 0.0901M_y$$
$$M_{y'} = -0.0901M_x + 0.9959M_y$$

Therefore

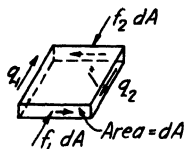
$$f = 0.0235M_x - 0.0128M_y$$

The same result is obtained directly from equation 11.27 by substituting the numerical values of I_x, I_y, and I_{xy}.

11.14 Shearing Stresses in Closed Section due to Transverse Forces

From the equilibrium conditions for an element *abcd* (Fig. 11.15), it is apparent that any variation in the normal stresses f due to a change in the bending moment must cause a change in the shearing stress. Thus, if f_1 and f_2 are the normal stresses, and q_1 and q_2 are the shearing forces per unit length, for an element of unit length

$$\sum F_z = 0$$

or

$$(f_1 - f_2)\,dA + q_1 - q_2 = 0$$
$$q_2 = q_1 + (f_1 - f_2)\,dA$$

$f_2\,dA$ q_1 q_2 $f_1\,dA$ Area = dA

Fig. 11.15

But

$$f_1 - f_2 = \frac{\Delta M_{x'} y'}{I_{x'}} + \frac{\Delta M_{y'} x'}{I_{y'}} = \frac{V_{y'} y'}{I_{x'}} + \frac{V_{x'} x'}{I_{y'}}$$

where $\Delta M_{x'}$ per unit length $= V_{y'} \times 1$
$\Delta M_{y'}$ per unit length $= V_{x'} \times 1$

$V_{y'}$ and $V_{x'}$ being the transverse shear parallel to the y' and x' axes, respectively.

Let

$$Q_{x'} = y'\,dA$$
$$Q_{y'} = x'\,dA$$

then

$$q_2 = q_1 + \frac{V_{y'}Q_{x'}}{I_{x'}} + \frac{V_{x'}Q_{y'}}{I_{y'}} \tag{11.28}$$

Equation 11.28 can be stated in the following terms: the unit shearing force q_2 at any point 2 is equal to the unit shearing force q_1 at any point 1 plus the change in the unit shearing force VQ/I between the two points with respect to the principal axes. If some point where the shearing stress is zero is known, such as at a free surface or an axis of symmetry, then all shearing stresses can be calculated from the change in the unit shearing force

$$q = \frac{V_{y'}Q_{x'}}{I_{x'}} + \frac{V_{x'}Q_{y'}}{I_{y'}} \tag{11.29}$$

However, for a closed section, such as Fig. 11.14, there are no points at which the shearing stresses are known, and consequently, there is one redundant q value in each cell. If the unit shearing force in the sheet between the flange areas A_2 and A_3 (Fig. 11.14) is designated by q' and the unit shearing force in the nose section by q'', then the unit shearing force q in any part of the perimeter can be expressed in terms of q' and q'' by equation 11.28. The two redundant quantities q' and q'' can then be determined from the condition that no twisting of the section takes place, in other words, that the angle of twist ϕ for each cell is zero. This strain condition gives for each cell in the section an equation of the type

$$\phi = \frac{1}{2AG}\sum q\frac{\Delta s}{t} = 0$$

or simply

$$\sum\frac{q\,\Delta s}{t} = 0 \tag{11.30}$$

By applying equation 11.30 to each cell, the two redundant quantities q' and q'' can be calculated. With q' and q'' known, the shearing force q at any part of the perimeter is determined from equation 11.28. This method of solution is applicable to any number of cells. The numerical operations will be explained by an example.

Example 11.6 The shear flow around the section shown in Fig. 11.14 will be determined for a transverse shear $V_{y'}$ of 1000 lb parallel to the

y' axis. The unit shearing force q' between the flange areas A_2 and A_3 and q'' in the nose section are selected as the redundant quantities. Using the value of I_x' that was computed in Example 11.5, the unit shearing force q at any part of the perimeter in the rear cell is given by equation 11.28 as

$$q = q' + \frac{1000Ay'}{223.7}$$

Since the flange areas are considered concentrated at certain points, the shear flow q is constant between these areas. Table 11.5 gives the

Table 11.5

Segment	Flange Area	y'	$Q = Ay'$	$\Delta q = \frac{1000Q}{223.7}$	$q = q' + \Delta q$
A_2–A_3	0.6	+6.34	+3.80	16.99	q'
A_3–A_4	0.4	+6.91	2.76	12.34	$q' + 12.34$
A_4–A_5	0.6	+6.32	3.79	16.94	$q' + 29.28$
A_5–A_6	0.6	−5.13	−3.08	−13.77	$q' + 15.51$
A_6–A_7	0.4	−7.73	−3.09	−13.81	$q' + 1.7$
A_7–A_8	0.4	−8.60	−3.44	−15.38	$q' - 13.68$
A_8–A_1 (web)	0.8	−8.45	−6.76	−30.22	$q' - q'' - 43.90$
A_1–A_2	1.2	+5.00	+6.00	+26.82	$q' - 17.08$

values of q for each segment of the perimeter between the flange areas. The calculations were made by starting with q' between areas A_2 and A_3 and progressing clockwise around the cell.

When the area A_8 is reached at the junction of the two cells, the equilibrium condition requires that the shear flow out of the junction must equal the shear flow in plus the change in the unit shear −30.22. The shear flow in the web between A_8 and A_1 is therefore

$$q = q' - 13.68 - q'' - 30.22 = q' - q'' - 43.9$$

In the same manner, the shear flow between areas A_1 and A_2 is

$$q = (q' - q'' - 43.9) + q'' + 26.82 = q' - 17.08$$

To evaluate q' and q'' the summation of the quantities $q\,\Delta s/t$ must be made zero for each cell. Table 11.6 gives the values of these quantities for the segments of the rear cell.

Table 11.6

Segment	q	Δs	t	$\frac{\Delta s}{t}$	$\frac{q\,\Delta s}{t}$
A_2–A_3	q'	10.9	0.072	151.3	$151.3q'$
A_3–A_4	$q' + 12.34$	14.6	0.072	202.7	$202.7q' + 2501.3$
A_4–A_5	$q' + 29.28$	11.5	0.102	112.7	$112.7q' + 3299.9$
A_5–A_6	$q' + 15.51$	14.6	0.051	286.2	$286.2q' + 4439.0$
A_6–A_7	$q' + 1.7$	10.9	0.051	213.6	$213.6q' + 363.1$
A_7–A_8	$q' - 13.68$	7.3	0.051	143.1	$143.1q' - 1957.6$
A_8–A_1	$q' - q'' - 43.90$	13.5	0.102	132.4	$132.4q' - 5812.4 - 132.4q''$
A_1–A_2	$q' - 17.08$	7.3	0.072	101.4	$101.4q' - 1731.9$
				For rear cell $\Sigma \frac{q\,\Delta s}{t}$	$= 1343.4q' - 132.4q'' + 1101.4$

$$\frac{q\,\Delta s}{t} \quad \text{for the nose element} = \frac{25q''}{0.051} = 490.2q''$$

For the web element $= -132.4q' + 132.4q'' + 5812.4$ or for the front cell

$$\sum \frac{q\,\Delta s}{t} = -132.4q' + 622.6q'' + 5812.4$$

Equating the summation of the $q\,\Delta s/t$ terms for each cell to zero gives

$$1343.4q' - 132.4q'' + 1101.4 = 0$$

$$-132.4q' + 622.6q'' + 5812.4 = 0$$

from which we obtain

$$q' = -1.76 \text{ lb per inch} \qquad q'' = -9.72 \text{ lb per inch}$$

11.15 Shear Center for Closed Section

In the preceding discussion of the shear flow in a closed section due to transverse shear, the location of the resultant applied shearing force was not specified. Because the determination of the redundant internal unit shearing forces is based on the assumption that there is no twist of the cross section due to the bending moments and transverse shears, the external shearing forces must be applied at that point in the cross section about which the resultant internal twisting moment of the shearing forces around the perimeter is zero. In algebraic terms this condition can be expressed as

$$\sum (q\,\Delta s)r' = 0 \tag{11.31}$$

where r' is the arm of the internal shearing force with respect to the shear center O'.

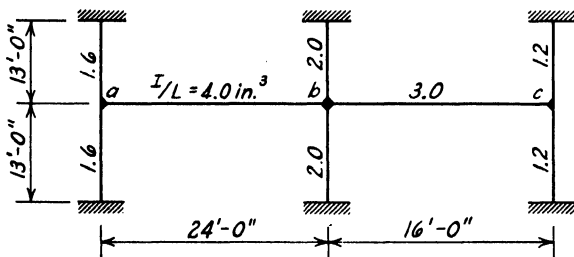
The position of the shear center O' (Fig. 11.14) is most easily obtained from the centroid O by the equation

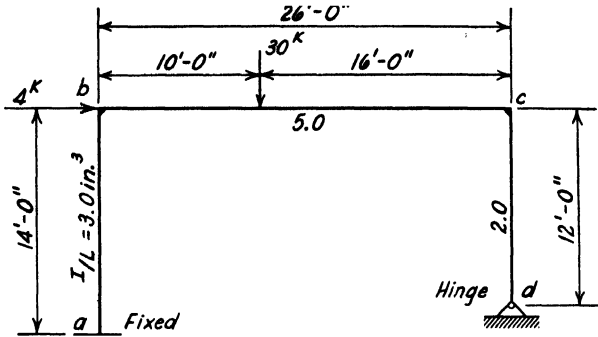
$$V\bar{r} = \sum (q\,\Delta s)r \tag{11.32}$$

in which r and $\bar{r}$ are measured with respect to the centroid O.

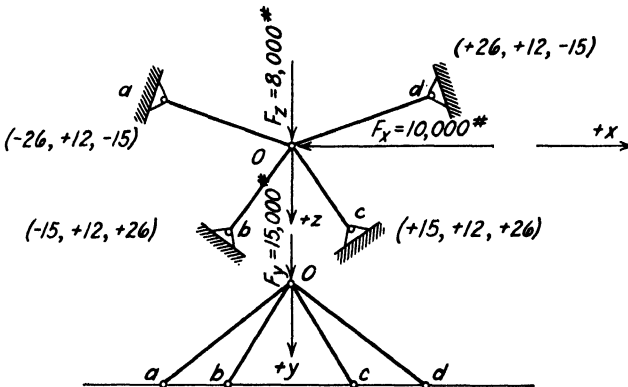
The transverse shear can first be taken parallel to the Y' axis as in Example 11.6, and the distance $\bar{r}_1$ can be determined. The calculations should then be repeated for a shear parallel to the X' axis and the distance $\bar{r}_2$ should be determined. It follows from the definition of the shear center that any system of applied forces can be resolved into resultant shearing forces through the shear center, a resultant torque, and bending moments about the principal axes. The stresses produced by these resultant forces can be analyzed independently in the manner explained in this section unless certain restrictions to the distribution of the stresses are imposed by the reactions or supports. The references should be consulted for a discussion of special problems that arise when the usual shearing deformation cannot take place.

Problems

11.1 Compare the maximum positive and negative bending moments in beam *ab* of the frame shown if all beam connections to the columns are first assumed rigid and then semirigid with a ψ value of 300 × 10^6 in.-lb. Use the following data: D. L. = 1.6 kips/ft, L. L. = 2.4 kips/ft, E = 29 × 10^6 psi. The I/L values in in.^{3} are recorded on the diagram. Assume the columns to be fixed at the floors above and below and neglect sidesway.

Problem 11.1

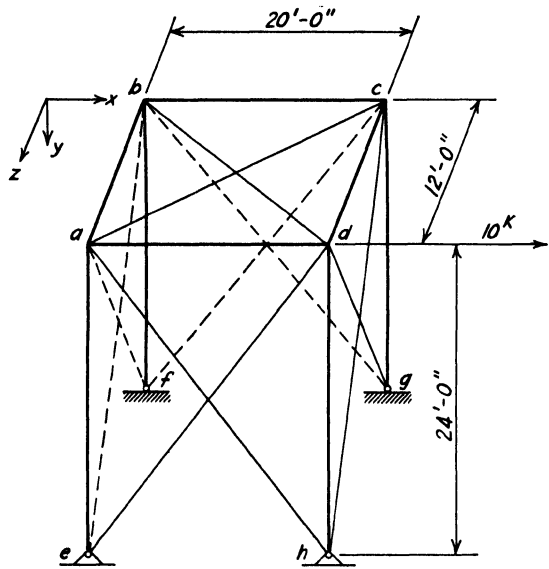
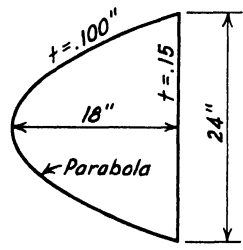
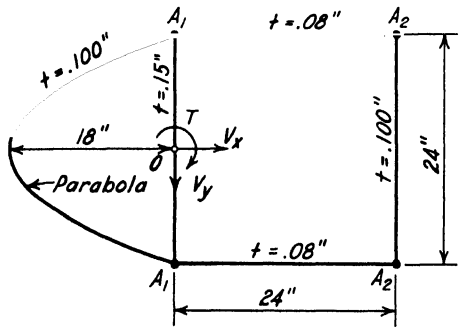
11.2 Calculate all end couples and the horizontal displacement of point *b* if the beam *bc* has semirigid connections with a ψ value of 400 × 10^6 in-lb. Use E = 29 × 10^6 psi and the I/L values in in^3. recorded on the diagram. Compare your values with those for rigid beam to column connections.

Problem 11.2

11.3 Solve Problem 11.2 using a ψ value of 200×10^6 in-lb. for the beam connections.

11.4 If a constant Q value of unity is assumed for all members of the space frame, determine the components of stress S_x, S_y, and S_z in each member by the deformation equations. Note the positive direction of the axes and the coordinates of all points with respect to point O.

Problem 11.4

11.5 Calculate the stresses for all members of the frame for the following areas: verticals 6 in.^{2}; horizontals 4 in.^{2}; diagonals 2 in.^{2}

11.6 Determine the maximum torque and angle of twist for the aluminum section in the diagram if the maximum allowable shearing stress in any element is 1200 psi. Use G = 3,800,000 psi.

11.7 What is the shearing stress in each element of the two-cell member if it is subjected to a torque of 30,000 ft-lb only?

Problem 11.5

Problem 11.6

Problem 11.7

11.8 Determine the shearing stresses in each plate element of the section in Problem 11.7 for a vertical shear $V_y = 8000$ lb, $V_x = 3000$ lb, and $T =$ 6000 ft-lb. taken with respect to point O. Assume that areas $A_1 = 1.8$ in.^{2} and $A_2 = 1.2$ in.^{2} take all normal stresses and that the plate elements have only shearing stresses acting on the right sections.

References

1 J. Charles Rathbun, "Elastic Properties of Riveted Connections," *Trans. Am. Soc. C. E.*, Vol. 101 (1936).

2 Bruce Johnston and E. H. Mount, "Analysis of Building Frames with Semi-Rigid Connections," *Trans. Am. Soc. C. E.*, Vol. 107 (1942).

3 Bruce Johnston and Robert Hechtman, "Design Economy by Connection Restraint," *Eng. News-Record* (October 10, 1940).

4 "First, Second and Final Reports of the Steel Structures Research Committee," Department of Scientific and Industrial Research, Great Britain, 1931–1936.

5 Inge Lyse and G. H. Gibson, "Welded Beam-Column Connections." *Am. Welding Soc.*, Vol. 15, pp. 34–40 (1936), and Vol. 16, pp. 2–9 (1937).

6 L. C. Maugh "Comments on Semi-Rigid Connections in Steel Frames," Final Report, Sixth Congress I.A.B.S.E. (1960).

7 R. V. Southwell, *Relaxation Methods in Engineering Science*, Oxford University Press, 1940.

8 K. Arnstein and E. L. Shaw, "On Methods of Calculating Stresses in the Hulls of Rigid Airships," Fifth Int. Congress for Applied Mechanics, 1938.

9 L. H. Donnell, H. B. Gibbons, and E. L. Shaw, "Analysis of Spoked Rings," Suppl. I, Report 2 of Special Committee on Airships.

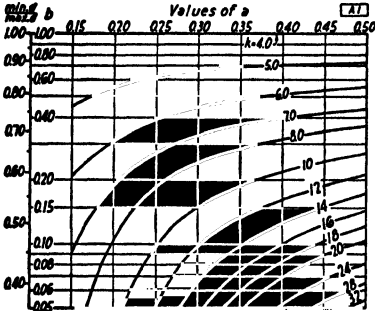
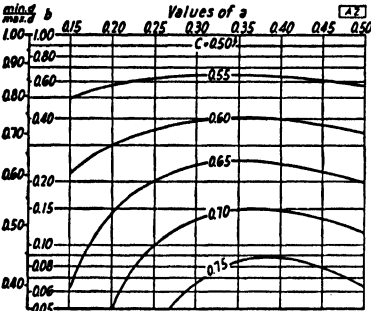
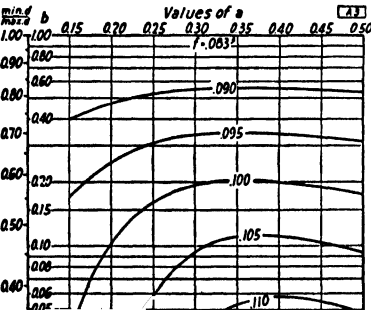
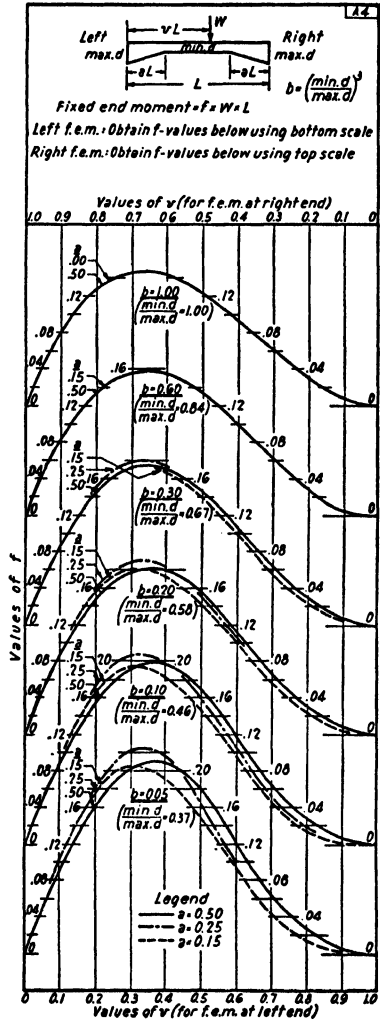
10 George F. Swain, *Structural Engineering*, McGraw-Hill Book Co.

11 David J. Peery *Aircraft Structures*, McGraw-Hill Book Co.

12 Th. von Karman and N. B. Christensen, "Methods of Analysis for Torsion with Variable Twist," *J. Inst. Aero. Sci.*, Vol. 11, p. 110 (1944).

13 J. E. Lothers, "Elastic Restraint Equations for Semi-Rigid Connections," *Trans. A.S.C.E.*, Vol. 116 (1951).

Appendix

Coefficients and Fixed-End Moments for Symmetrical Beams with Straight Haunches

Courtesy of the Portland Cement Association.

Appendix

Coefficients and Fixed-End Moments for Symmetrical Beams with Parabolic Haunches

Courtesy of the Portland Cement Association.

Appendix

Coefficients and Fixed-End Moments at the Large End of an Unsymmetrical Beam with a Straight Haunch

Courtesy of the Portland Cement Association.

Appendix

Coefficients and Fixed-End Moments at the Small End of an Unsymmetrical Beam with a Straight Haunch

Courtesy of the Portland Cement Association

Appendix

Coefficients and Fixed-End Moments at the Large End of an Unsymmetrical Beam with a Parabolic Haunch

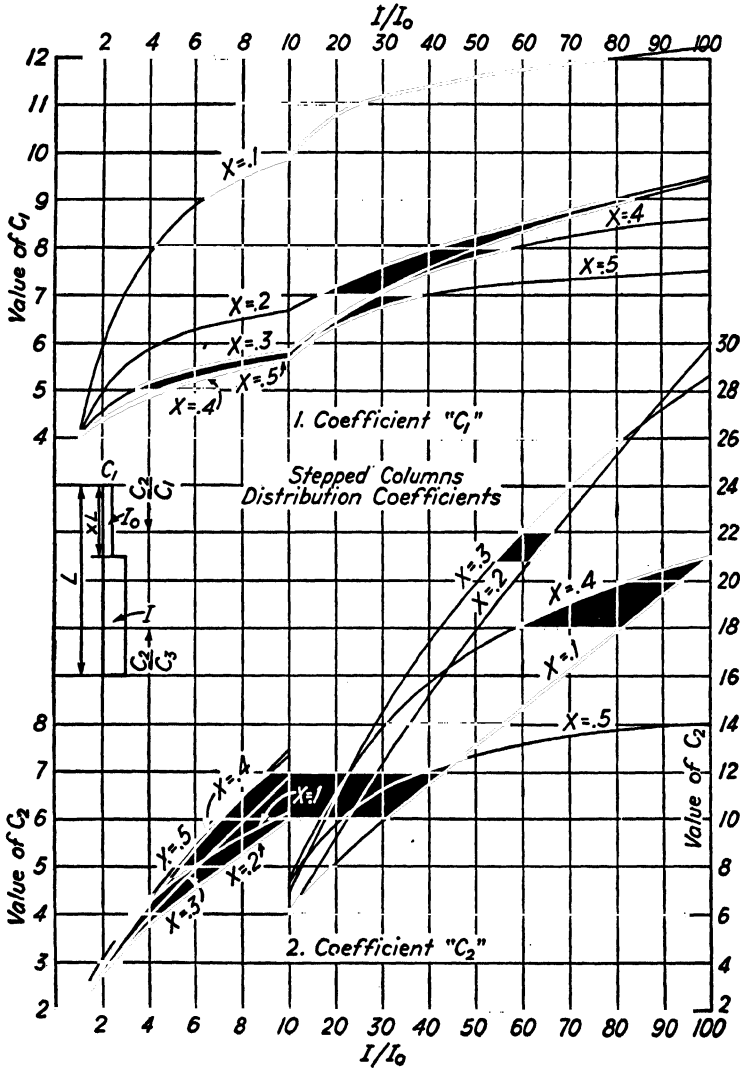
Courtesy of the Portland Cement Association

Appendix

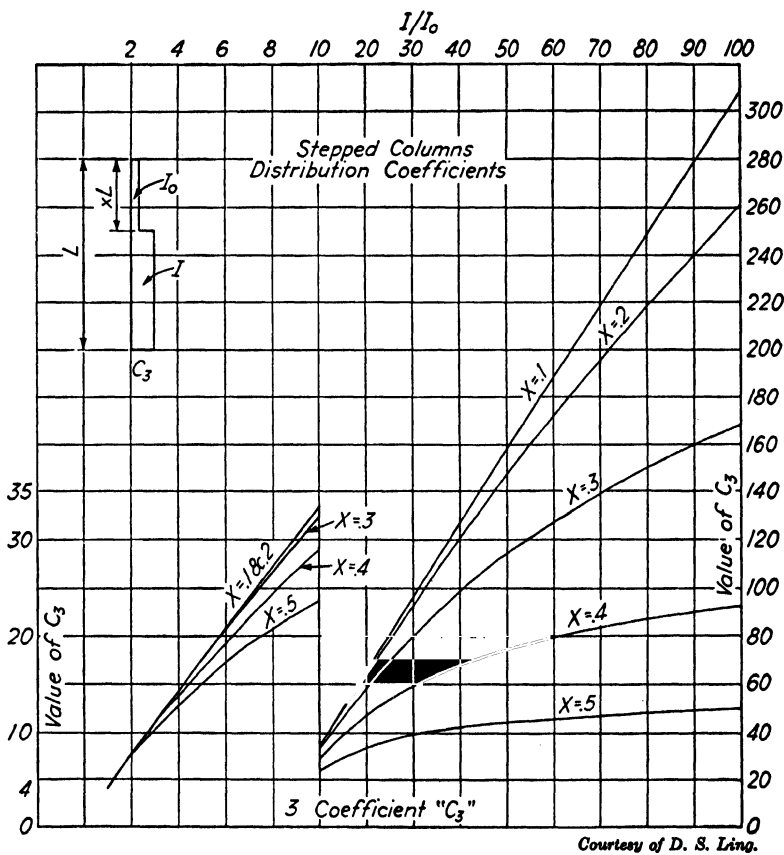
Coefficients and Fixed-End Moments at the Small End of an Unsymmetrical Beam with a Parabolic Haunch

Courtesy of the Portland Cement Association.

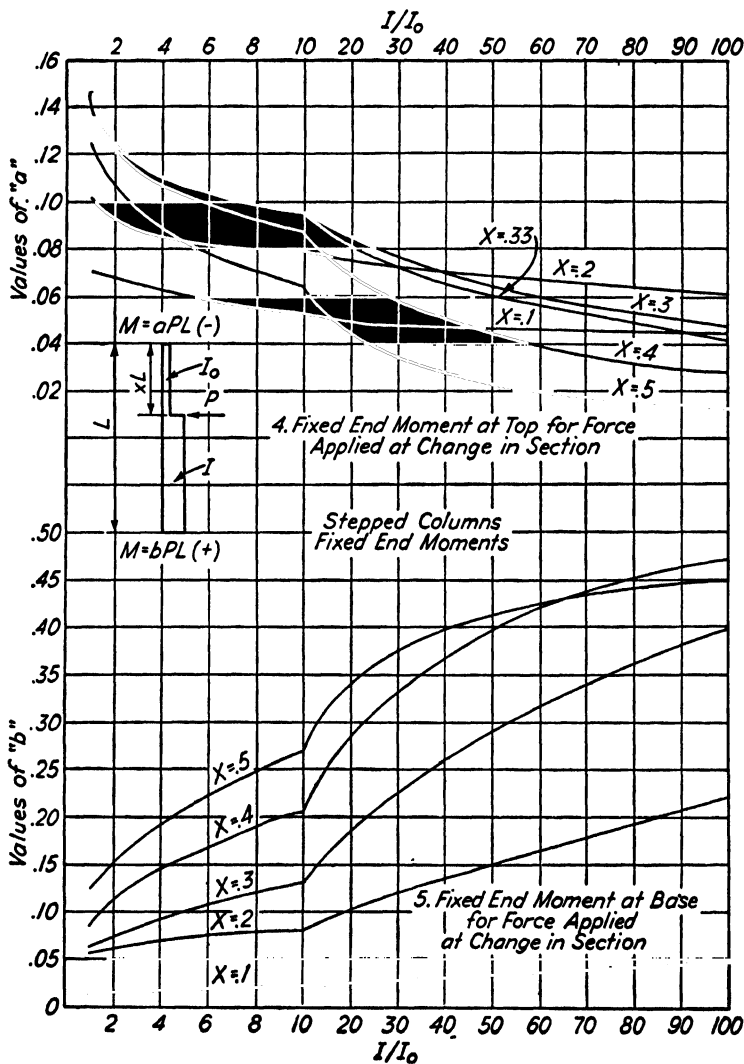
Appendix

Coefficients C_1 and C_2 for Beams and Columns with Sudden Change in Cross Section

Courtesy of D. S. Ling.

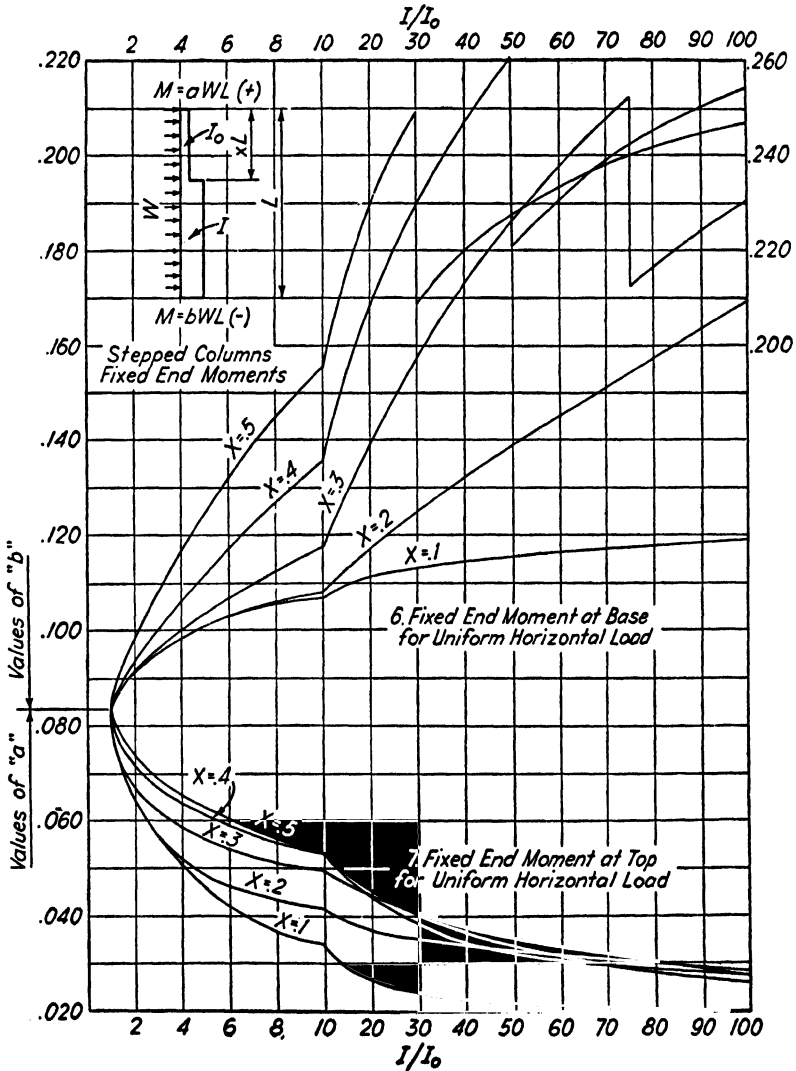
Appendix

Coefficients C_3 for Beams and Columns with Sudden Change in Cross Section

Courtesy of D. S. Ling.

Appendix

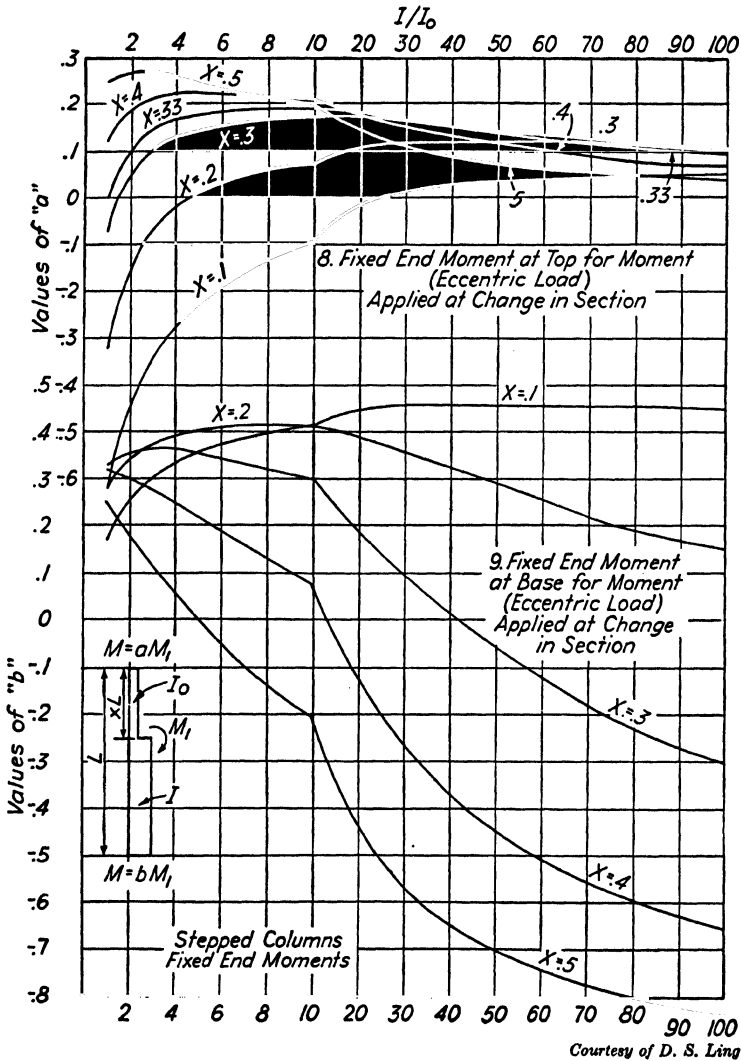
Fixed-End Moments for Beams and Columns with Concentrated Load Applied at Change in Cross Section

Courtesy of D. S. Ling.

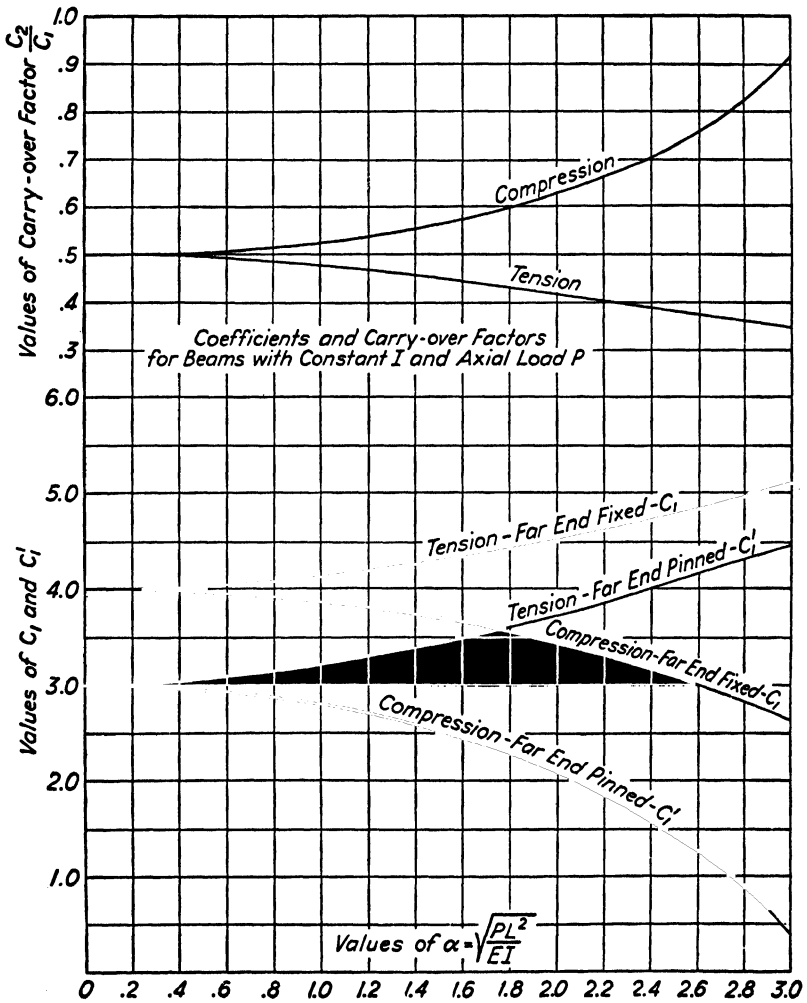
Appendix

Fixed-End Moments for a Uniform Load Acting on Beams and Columns with Sudden Change in Cross Section

Courtesy of D. S. Ling.

Appendix

Fixed-End Moments for Beams and Columns with an External Moment Applied at Change in Cross Section

Courtesy of D. S. Ling

Appendix

Coefficients for Members with Axial Load and Constant Cross Section

Appendix

Fixed-End Moments for a Uniform Load Applied to Members with Axial Load and Constant Cross Section

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